

EXAMPLE SHEET 2

Examples Class 2: Friday November 15, 1:30-3:30 in MR 9. I will mark problems 2, 5, and 10. Hand work in by 5pm on November 13 if you would like it marked. Questions 13-15 are optional. They are designed to illustrate some ideas relating to the Hurewicz and Whitehead theorems.

1. If $f_0, f_1 : S^{n-1} \rightarrow X$ and $f_0 \sim f_1$, show that $X \cup_{f_0} D^n \sim X \cup_{f_1} D^n$.
2. Consider the cell structure on S^n which has two cells of dimension k for each $0 \leq k \leq n$, corresponding to the northern and southern hemispheres of $S^k \subset S^n$. Write out its cellular chain complex and verify that it has the correct homology.
3. Let $X = D^{n+1} \times S^m / S^n \times S^m$. Find a cell decomposition of X and compute its homology using the cellular chain complex.
4. Write $S^{2k-1} = \{(z_1, \dots, z_k) \in \mathbb{C}^k \mid \sum |z_i|^2 = 1\}$. The group \mathbb{Z}/p acts on S^{2k-1} by $a \cdot \mathbf{z} = \lambda^a \mathbf{z}$, where $\lambda = e^{2\pi i/p}$. The *lens space* L_p^{2k-1} is the quotient $S^{2k-1}/(\mathbb{Z}/p)$. Show that L_p^{2k-1} has a cell decomposition with one cell of each dimension between 0 and k . Find $C_*^{cell}(L_p^{2k-1})$ and compute its homology.
5. Suppose G_i ($1 \leq i \leq n$) are finitely generated abelian groups. Show that there is a finite cell complex X with $H_i(X) = G_i$ for $1 \leq i \leq n$ and $H_i(X) = 0$ for $i > n$.
6. Suppose $x \in H_n(X)$, where X is an arbitrary topological space. Show that there is a finite cell complex A and a map $f : A \rightarrow X$ so that $x \in \text{im } f_*$.
7. Let X and Y be finite cell complexes. A map $f : X \rightarrow Y$ is cellular if $f(X_n) \subset Y_n$ for all n . Show that such an f induces a chain map $f_*^{cell} : C_*^{cell}(X) \rightarrow C_*^{cell}(Y)$, and that the induced map on homology agrees with f_* .
8. Let $f : S^n \rightarrow S^n$ have degree k , and let $X = S^n \cup_f D^{n+1}$. Let $\pi : X \rightarrow X/S^n \simeq S^{n+1}$ be the quotient map. Compute $\pi^* : H^*(S^{n+1}) \rightarrow H^*(X)$ and $\pi_* : H_*(X) \rightarrow H_*(S^{n+1})$.
9. Let $\partial : H_*(X, A) \rightarrow H_{*-1}(A)$ and $\delta : H^*(A) \rightarrow H^{*+1}(X, A)$ be the boundary maps in the long exact sequences of the pair (X, A) . If $a \in H^k(A)$ and $x \in H_{*+1}(X, A)$, show that $\langle a, \partial x \rangle = \langle \delta a, x \rangle$.
10. Suppose $0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$ is a short exact sequence of abelian groups. If X is a space, show that there is a short exact sequence

$$0 \rightarrow C_*(X; K) \rightarrow C_*(X; G) \rightarrow C_*(X; H) \rightarrow 0.$$

Let $\beta_X : H_*(X; H) \rightarrow H_{*-1}(X; K)$ be the boundary map in the associated exact sequence. (β_X is called the *Bockstein homomorphism*.) If $f : X \rightarrow Y$, show that $f_*\beta_X = \beta_Y f_*$. If $X = Y = \mathbb{RP}^n$, deduce that $f_* : H_*(\mathbb{RP}^n; \mathbb{Z}/2) \rightarrow H_*(\mathbb{RP}^n; \mathbb{Z}/2)$ is determined by $f_* : H_*(\mathbb{RP}^n; \mathbb{Z}) \rightarrow H_*(\mathbb{RP}^n; \mathbb{Z})$. Show that the converse holds if n is even, but not if n is odd.

11. Let C_* be a free finitely generated chain complex defined over \mathbb{Z} . Show that $H_*(C) = 0$ if and only if $H_*(C \otimes \mathbb{Z}/p) = 0$ for every prime p . Now suppose $f : C_* \rightarrow C'_*$ is a chain map. Using problem 5 on Example Sheet 1, show that f is a chain homotopy equivalence if and only if the map $f_* : H_*(C \otimes \mathbb{Z}/p) \rightarrow H_*(C' \otimes \mathbb{Z}/p)$ is an isomorphism for every prime p .
12. Let $R = \mathbb{C}[x]/(x^3)$, and for $i = 1, 2$, let M_i be the R -module $\mathbb{C}[x]/(x^i)$. Find a free resolution of M_1 and use it to compute $\text{Tor}_*^R(M_1, M_1)$ and $\text{Tor}_*^R(M_1, M_2)$.
13. * Let X be a finite cell complex, and let A be a subcomplex of X . Let $Y = X \times \{0\} \cup A \times I \subset X \times I$. Show that any map $Y \rightarrow Z$ extends to a map $X \times I \rightarrow Z$. (Hint: start with the case $X = D^n$, $A = S^{n-1}$.)
14. * Let $f : X \rightarrow Y$ be a map of finite cell complexes, and suppose $\pi_i(Y, *) = 0$ for all $0 < i \leq n$. Using problem 10, show there is a map $\bar{f} : X/X_n \rightarrow Y$ so that $\bar{f} \circ \pi \sim f$, where $\pi : X \rightarrow X/X_n$ is the quotient map. Deduce the following consequences:
 - (a) $H_i(Y) = 0$ for $0 < i \leq n$.
 - (b) The Hurewicz homomorphism $\psi : \pi_{n+1}(Y, *) \rightarrow H_{n+1}(Y)$ is surjective.
 - (c) If $\pi_i(Y, *) = 0$ for all $i > 0$, then Y is contractible.
15. * This question assumes some knowledge of π_1 .
 - (a) Construct a connected finite cell complex X with $\tilde{H}_*(X) = 0$ but which is not contractible.
 - (b) Find a map $f : S^1 \vee S^2 \rightarrow S^1 \vee S^2$ which induces the identity map on $H_*(S^1 \vee S^2)$, but is not homotopic to the identity.

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