Algebraic Topology Part III, 2018-19: Sheet 1

1. (a) Prove homotopy equivalence is an equivalence relation on topological spaces.

   (b) Which of the following are homotopy equivalent to S^1? (i) the annulus $\{1 < |z| < r\}$ (ii) a bagel (iii) a genus two surface with a disc sewn across one of the holes (iv) a giraffe (v) the complement of a point in the real projective plane $\mathbb{RP}^2$.

2. Compute $H^0(X;\mathbb{Z})$ for a topological space X. Give an example of a space X for which $H_0(X;\mathbb{Z})$ and $H^0(X;\mathbb{Z})$ are not isomorphic.

3. What can you say about the group G and/or the homomorphism α in an exact sequence of the shape

   (a) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$;

   (b) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$;

   (c) $0 \to \mathbb{Z}/4 \xrightarrow{\alpha} G \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$?

4. (a) The *suspension* ΣX of a space X is the quotient of $X \times [0,1]$ by the map which collapses each end of the cylinder to a point: $X \times \{0\} \simeq p_0$ and $X \times \{1\} \simeq p_1$. Observe $\Sigma S^n = S^{n+1}$. Hence or otherwise prove there are maps $f : S^n \to S^n$ of any degree, for any $n > 0$.

   (b) Suppose A is a closed connected manifold. Is ΣA necessarily homeomorphic to a closed manifold? Justify your answer.

5. (a) Compute the homology groups of the closed orientable surface Σ_g of genus g.

   (b) Compute $H_*(\Sigma_2, A;\mathbb{Z})$ where A is a simple closed curve which: (i) separates Σ_2 into two genus one pieces with one boundary component each; (ii) is a non-separating simple closed curve cutting along which gives a genus one surface with two holes, and (iii) bounds an embedded disc.

6. Using Mayer-Vietoris, compute the cohomology groups of complex projective space $\mathbb{CP}^k$. For each n, construct a closed connected four-dimensional manifold X_n with $H^1(X_n) = 0$ and $H^2(X_n) \cong \mathbb{Z}^n$. [*Hint: look up the "connect sum".*]

7. (a) Define relative cohomology $H^*(X,A)$ in such a way that there is a long exact sequence
$$\cdots \to H^i(X,A) \to H^i(X) \to H^i(A) \to H^{i+1}(X,A) \to \cdots$$

   (b) Compute the relative cohomology $H^*(D, \{p_1,\ldots,p_k\};\mathbb{Z})$ of the closed disc in $\mathbb{C}$ relative to k points.

8. Prove that the long exact sequence associated to a short exact sequence of chain complexes is really exact. Feel reassured by my decency in not going through the proof in lectures.

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