Part III Algebraic Geometry

Example Sheet IV, 2021.

Note: If you would like to receive feedback, please turn in solutions to any subset of Questions 1,4,7,8. You may leave your solutions with my pigeon in the CMS, or send it to me via the interweb by any time before the fourth examples class.

1. Let D be a degree 0 Weil divisor on $\mathbb{P}^1_k$. B Construct a rational function f, i.e. a section of the structure sheaf over the generic point, such that the divisor associated to f is D. Deduce that every degree 0 divisor on $\mathbb{P}^1_k$ is principal.

2. A fact from the theory of varieties is that if E is a smooth projective curve of genus 1, then E is not isomorphic to $\mathbb{P}^1$. Let p and q be distinct points on E. By using this fact, show that the divisor $p - q$ is a non-principal degree 0 divisor on E.

3. In lectures, we sketched the construction of a line bundle associated to a Cartier divisor. You may review this construction by examining the two definitions before Propositions 6.11 and 6.13 in Chapter II of Hartshorne. Let H be any hyperplane in $\mathbb{P}^n_k$. Prove that the sheaf associated to the Cartier divisor $[H]$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$, defined earlier via the Proj construction for a graded module[1].

4. Let X be a Noetherian scheme and $\mathcal{A}_\bullet$ be a coherent sheaf of $\mathcal{O}_X$ algebras. We make the simplifying assumption that $\mathcal{A}_1$ is coherent and that $\mathcal{A}_1$ locally generates the algebra $\mathcal{A}_\bullet$ [i.e. locally, the algebra of sections over an open is generated in degree 1]. By direct analogy with the construction of global Spec on the previous example sheet, construct global Proj
$$\pi : \underline{\mathrm{Proj}}\,\mathcal{A}_\bullet \to X.$$
Observe that the fibres of π can be described by using the usual Proj construction. Construct the global analogue of the sheaf $\mathcal{O}(1)$, i.e. construct a line bundle on $\mathcal{L}$ on $\underline{\mathrm{Proj}}\,\mathcal{A}_\bullet$ such that, for each point x of X, the restriction of $\mathcal{L}$ to the fibre $\pi^{-1}(x)$ is the "usual" $\mathcal{O}(1)$ on that fibre.

5. Let X be a noetherian scheme as above and let $\mathcal{A}_\bullet = \mathcal{O}_X[T_0, \dots T_n]$ be the sheaf of algebras whose value over an affine U is a polynomial algebra on $n+1$ variables over the ring of functions on U. Give it a grading by requiring each variable to have weight 1. Show that the global Proj applied to $\mathcal{A}_\bullet$ gives
$$\underline{\mathrm{Proj}}\,\mathcal{O}_X[T_0, \dots T_n] = \mathbb{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} X.$$
This is the most boring instance of the global Proj construction.

6. Let X be Noetherian and let $\mathcal{I}$ be a coherent sheaf of ideals. Consider the graded algebra
$$\mathcal{A}_\bullet := \bigoplus_{d \geq 0} \mathcal{I}^d.$$
This algebra satisfies the simplifying assumption in Question 5, so we can apply global Proj. Define the *blowup of X along $\mathcal{I}$* to be
$$\mathrm{Bl}_{\mathcal{I}} X := \underline{\mathrm{Proj}}\,\mathcal{A}_\bullet.$$
To start, make sense of this construction. If X is integral and $\mathcal{I}$ is not the unit ideal, show that the natural morphism
$$\mathrm{Bl}_{\mathcal{I}} X \to X$$
is an isomorphism on a non-empty (in fact, dense) open set.

7. (The blowup of affine space; hard but crucial) Let A be the ring $k[X_1, \dots, X_n]$ and let I be the ideal $\langle X_1, \dots, X_n \rangle$. By using the surjection
$$A[Y_1, \dots, Y_n] \to \bigoplus_{d \geq 0} I^d$$
sending Y_i to X_i, identify the blowup of $\mathbb{A}^n_k$ at I with a closed subscheme in $\mathbb{P}^{n-1}_A$. Let $n = 2$ and let X denote the blowup above in this case. Show that the fiber of
$$X \to \mathbb{A}^2_k$$
over the point $(0, 0)$ is naturally identified with $\mathbb{P}^1$.

[1] Beware that sign conventions vary when it comes to $\mathcal{O}(1)$ and $\mathcal{O}(-1)$, but only one will make any sense given the context, so it's not a serious source of confusion.

8. Let $X = \operatorname{Spec} A$ be affine, and let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$
is exact. [You may freely use the following fact about quasi-coherent sheaves: if $\mathcal{F}$ is a quasi-coherent sheaf on a scheme X, and $U \subseteq X$ is open affine, then $\mathcal{F}|_U$ can be written as a cokernel of a morphism between free sheaves on U. This is an analogue of the type of result concerning locally Noetherian schemes and locally finite type morphisms. These definitions state the existence of an affine open cover with certain properties and then one shows that the required properties hold for any affine open. You may find a proof of this fact in Hartshorne II, §5.]

9. Prove that an injective sheaf of abelian groups on a topological space is flasque, i.e. all restriction maps are surjective. [This has very googleable answer, so try it first before doing that.]

10. Let X be the closed subscheme of $\mathbb{P}^2_k$ defined by the scheme theoretic vanishing locus of a homogeneous polynomial of degree d. Assume that $(1,0,0) \notin X$. Then show X can be covered by the two affine open subsets $U = X \cap D_+(x_1)$, $V = X \cap D_+(x_2)$. Now compute the Čech complex explicitly and show that
$$\begin{aligned} \dim H^0(X, \mathcal{O}_X) &= 1 \\ \dim H^1(X, \mathcal{O}_X) &= (d-1)(d-2)/2 \end{aligned}$$
where d is the degree of f. Compare with the degree-genus formula for a plane curve.

11. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, $U = X \setminus \{(x,y)\}$ (removing the maximal ideal corresponding to the origin). By choosing a suitable affine cover of U, show that $H^1(U, \mathcal{O}_U)$ is naturally isomorphic to the infinite dimensional k-vector space with basis $\{x^i y^j \mid i, j < 0\}$. Thus in particular U is not affine.