Part III Algebraic Geometry

Example Sheet III, 2021.

Note: If you would like to receive feedback, please turn in solutions to any subset of Questions 1,3,5,8. You may leave your solutions with my pigeon in the CMS by any time on November 24 2021.

1. Let Z be the closed subscheme of $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$ given by the vanishing locus of the homogeneous ideal $(XY - tZ^2)$, where X, Y, Z are the homogeneous coordinate functions on $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$. (i) Calculate[1] the scheme theoretic fibers of Z over each of the two points of $\operatorname{Spec}\mathbb{C}[\![t]\!]$. (ii) For each integer $n \geq 1$, inductively define Z^n to be the fiber product of Z with Z^{n-1} over $\operatorname{Spec}\mathbb{C}[\![t]\!]$. Calculate the fiber of Z^n over the closed point of $\operatorname{Spec}\mathbb{C}[\![t]\!]$. Calculate the number of irreducible components of this closed fiber as a function of n.

2. By using the valuative criterion of properness, verify the following statements about morphisms between Noetherian schemes. (i) A closed immersion is proper. (ii) A composition of proper morphisms is proper. (iii) Properness is stable under base change.

3. In lecture, we have claimed that given a module M over a ring A, there is an associated sheaf M^{sh} on $\operatorname{Spec}(A)$ whose value over a distinguished open U_f is identified with the localization of M at f. We have also claimed that if $A_\bullet$ is an $\mathbb{N}$-graded ring and $M_\bullet$ is a graded $A_\bullet$–module, there is an analogous sheaf on $\operatorname{Proj}(A_\bullet)$ whose sections, over a distinguished open associated to a positive degree element, are the degree 0 elements in the localization of $M_\bullet$ at f. Give precise definitions of these sheaves and verify that they do indeed form sheaves of modules over the structure sheaf.

   Given a graded ring $A_\bullet$ and a graded module $M_\bullet$, define the sheaf $M_\bullet(1)$ to be graded module whose weight n piece is the weight $n+1$ piece of $M_\bullet$. Let X be $\operatorname{Proj}(A_\bullet)$. The sheaf associated to $A_\bullet(1)$ is denoted $\mathcal{O}_X(1)$. Take $A_\bullet$ to be $\mathbb{C}[x, y, z]$ with the standard grading. Calculate the sections of this sheaf over the standard distinguished open sets in $\mathbb{P}^2_{\mathbb{C}}$. Repeat the exercise for the sheaves $\mathcal{O}_X(d)$ for all integers d. For which d does $\mathcal{O}_{\mathbb{P}^2}(d)$ have non-zero global sections?

In the next two exercises, you will prove the valuative criterion for separatedness of a morphism $X \to Y$ of schemes, with X Noetherian. We start with some preparations.

4. A discrete valuation ring (DVR) is a local principal ideal domain. Below denote DVRs by R and their fraction fields by K. If A is a local ring, we use $\mathfrak{m}_A$ to denote its maximal ideal.

   (i) Prove that a DVR R has exactly two prime ideals. The open point of $\operatorname{Spec} R$ will be called the generic point, and the closed point will be called the closed point.

   (ii) Let X be any scheme. Prove that a morphism $\operatorname{Spec} R \to X$ is equivalent to giving two points p_0 and p_1 in X such that the closure of p_0 contains p_1, and the following ring theory data: an inclusion of fields $\kappa(p_0) \hookrightarrow K$ such that under this inclusion, the subring R contains the local ring of $\mathcal{O}_{Z,p_1}$, and the ideal $\mathfrak{m}_{\mathcal{O}_{Z,p_1}}$ is the intersection of $\mathfrak{m}_R$ with $\mathcal{O}_{Z,p_1}$.

5. In a topological space X, a point p_1 is a *specialization* of another point p_0 if p_1 is contained in the closure of p_0. A closed subset of X is certainly stable under specialization of points. We will use a standard fact (Hartshorne II.4.5) that if $f : X \to Y$ is a scheme map such that the preimage of every quasi-compact subset on Y is a quasi-compact subset of X, then closedness of the subset $f(X)$ is equivalent to $f(X)$ being stable under specialization[2].

   (i) Suppose $f : X \to Y$ is separated. Fix a map $\operatorname{Spec} R \to Y$, and a map $\operatorname{Spec} K \to X$ making the natural diagram commute. Suppose we are given two maps $\operatorname{Spec} R \to X$ compatible with the data we have fixed. Construct the natural map from $\operatorname{Spec} R$ to $X \times_Y X$ and use closedness of the diagonal to show that the two maps from $\operatorname{Spec} R$ to X must coincide.

   (ii) In order to show the converse, you must demonstrate that the image of X under the diagonal map $\Delta_{X/Y} : X \to X \times_Y X$ is stable under specialization. In order to do this, choose a point p_0 on the diagonal and a specialization p_1 of it in $X \times_Y X$. Use Part (ii) of the previous question to find a DVR R that maps to $X \times_Y X$, and use the two projection maps to X and separatedness of X to deduce that p_1 must lie in the diagonal.

[1] In this context, calculate means to describe the scheme up to isomorphism using basic operations: standard rings, Spec and Proj, and gluing constructions, as explicitly as possible.

[2] If you are unhappy taking this on faith, you should look at Chevallay's theorem showing that images of scheme maps are constructible and based on that, try to prove the claimed equivalence yourself.

6. Let X be a scheme, $f : \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that $\ker f, \mathrm{coker} f$ and $\mathrm{im} f$ are quasi-coherent. Further, if X is Noetherian and $\mathcal{F}$ and $\mathcal{G}$ are coherent, show $\ker f, \mathrm{coker} f$ and $\mathrm{im} f$ are coherent.

   Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_Y$-modules. Show that $f^*\mathcal{F}$ is a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules.

   Show by example that if $\mathcal{G}$ is a coherent sheaf on X, then $f_*\mathcal{G}$ need not be a coherent sheaf on Y. [Note: $f_*\mathcal{G}$ is always quasi-coherent, but this is harder to prove.]

7. Let $i : Z \to X$ be a closed immersion of schemes. Recall this means i is a homeomorphism of Z onto a closed subset of X, and the map $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. We write $\mathcal{I}_{Z/X} = \ker i^\#$.

   a) Show that $\mathcal{I}_{Z/X}$ is a *sheaf of ideals* of $\mathcal{O}_X$, i.e., $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ for each $U \subseteq X$ open.
   b) Show that $\mathcal{I}_{Z/X}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, and is coherent if X is Noetherian.
   c) Show that there is a one-to-one correspondence between quasi-coherent sheaves of ideals of X and closed subschemes of X.

8. (Global Spec). Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$–algebras on a scheme X. Assume that the sheaf is quasi-coherent. We will now construct the "relative" or "global" Zariski spectrum $\pi : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ as a scheme with a map to X.

   (Set) Given a point $p \in X$, the set $\pi^{-1}(p)$ is defined to be $\mathrm{Spec}\,(\mathcal{A} \otimes \kappa(p))$, where $\kappa(p)$ is the residue field of X at p. Ranging over all p in X defines the points of $\underline{\mathrm{Spec}}\,\mathcal{A}$ with a natural map to X.

   (Topology) Given an affine open U in X, describe a natural bijection between $\pi^{-1}(U)$ and the spectrum of the algebra $\mathcal{A}(U)$. Use this to upgrade $\pi : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ to a continuous map of topological spaces.

   (Functions) Construct a structure sheaf on $\underline{\mathrm{Spec}}\,\mathcal{A}$ by using the identification of sets of the form $\pi^{-1}(U)$ with the spectrum of $\mathcal{A}(U)$ to endow such sets with a natural ring of functions.

   Show that all these data produce a scheme map $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$.

   [If the sheaf of algebras is graded, one can similarly construct a global Proj scheme of X.]