Part III Algebraic Geometry

Example Sheet I, 2021

Note: If you would like to receive feedback, please turn in solutions to any subset of Questions 1,7,8,11,13 to the course instructor by email by Thursday, October 28.

1. Describe the topological spaces $\operatorname{Spec}\mathbb{R}[x]$, $\operatorname{Spec}\mathbb{C}[x,y]$, $\operatorname{Spec}\mathbb{Z}[x]$, and $\operatorname{Spec}\mathbb{C}[\![x]\!]$. In each case, describe the subset of maximal ideals.

2. Given an example of a homomorphism of rings $\varphi : A \to B$ such that the preimage of a maximal ideal is not maximal. Prove that if φ is surjective then the preimage of a maximal ideal is maximal.

3. Let X_1 and X_2 be the Zariski spectra of rings A_1 and A_2. Describe a natural ring whose Zariski spectrum is homeomorphic $X_1 \sqcup X_2$. Notice in particular how this is not $X_1 \times X_2$.

4. Find a ring A whose Zariski spectrum is the *connected doubleton*, i.e. the topological space consisting of two points $\{p_1, p_2\}$ such that $\{p_1\}$ is dense and $\{p_2\}$ is closed. ($\star$) Find a ring A whose Zariski spectrum consists of three points $\{q_1, q_2, q_3\}$ such that $\{q_1\}$ is dense, the set $\{q_2\}$ contains q_3 in its closure, and $\{q_3\}$ is closed.

5. Let A be the quotient of a polynomial ring in finitely many variables by a prime ideal. Let $m\operatorname{Spec}(A)$ be the set of maximal ideals of A equipped with the Zariski topology. Describe a procedure that reconstructs the full Zariski spectrum $\operatorname{Spec}(A)$ and its topology in terms of the irreducible closed subsets of $m\operatorname{Spec}(A)$. Apply this procedure with $A = \mathbb{C}[\![x]\!]$ to conclude that some rings "do not have enough maximal ideals".

6. (Sheafification is functorial) Prove that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there is an induced morphism $f^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$ with $(f^{\mathrm{sh}})_p = f_p$.

7. Describe a non-zero presheaf of abelian groups all of whose stalks are 0. Conclude that the sheafification is the constant sheaf 0.

8. (Exactness is stalk local) Show that a sequence $\cdots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$ is exact if and only if for every $p \in X$, the corresponding sequence of maps of abelian groups is exact.

9. Show that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism between sheaves, then the sheaf image $\operatorname{im} f$ can be naturally identified with a subsheaf of $\mathcal{G}$.

10. Show a morphism of sheaves is an isomorphism if and only if it is injective and surjective.

11. (f^{-1} and f_* are adjoint functors.) Given a continuous map $f : X \to Y$, sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, construct natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use this to construct a bijection

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}),$$

   (i.e., f^{-1} is left adjoint to f_* and f_* is right adjoint to f^{-1}.)

12. Observe that there is a unique morphism from $\mathbb{Z}$ to any commutative ring with identity, i.e. $\mathbb{Z}$ is an initial object in this category. Show that $\operatorname{Spec}\mathbb{Z}$ is a final object in the category of schemes, i.e., every scheme has a unique morphism to $\operatorname{Spec}\mathbb{Z}$.

13. (Gluing) Let $\{X_i\}$ be a family of schemes (possibly infinite) and suppose for each $i \neq j$ we are given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$, such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$ and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

   Then show there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i with an open subscheme of X; (2) the $\psi_i(X_i)$ cover X; (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$; and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.