III Algebra Michaelmas Term 2019

EXAMPLE SHEET 4

All rings are commutative with a 1 unless stated otherwise.

1. Let k be a field and f be a homogeneous polynomial of positive degree in the ring $R = k[X_1, \ldots, X_n]$, graded in the usual way. Calculate the Hilbert polynomial for the graded module $R/(f)$ and hence show that the degree of the Samuel function of $R/(f)$ with respect to the maximal ideal $(X_1, \ldots, X_n)$ is $n - 1$.

2. Let R be a Noetherian local ring with maximal ideal P. Show for non-zero-divisor x that $d(R/(x)) \leq d(R) - 1$.

3. Let R be a Noetherian local ring with maximal ideal P. Show that $\dim(R) \leq d(R)$. Furthermore suppose that R is a regular local ring. Show that $\dim(R) = d(R)$ and that the associated graded ring of R with respect to the P-adic filtration is isomorphic to a polynomial ring. Deduce that R is an integral domain.

4. Let R be a ring and let E be an R-module. Show that the following are equivalent. (1) E is injective; (2) If $\mu : E \longrightarrow M$ is a monomorphism then there exists $\beta : M \longrightarrow E$ such that $\beta\mu$ is the identity map; (3) E is a direct summand in every module which contains E as a submodule.

5. Le R be a ring. An R-module is said to be *divisible* if, for every e in E and every r in R which is not a zero-divisor, there exists e' in E such that $e = re'$. Show that an injective R-module is necessarily divisible.

6. Let R be a principal ideal domain. Show that an R-module is injective if and only if it is divisible.

7. Let R be the ring of integers. Show that any R-module may be embedded in an injective R-module. Let S be a ring and let M be an injective R-module. Show that $\mathrm{Hom}_R(S, M)$ is an injective S-module. Deduce that any S-module can be embedded in an injective S-module.

8. Let R be a ring and let I and J be ideals. Show that (a) $\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/IJ$, and (b) $\mathrm{Tor}_2(R/I, R/J) = \ker(I \otimes_R J \longrightarrow IJ)$

9. Let R be the ring of integers. Show that $\mathrm{Ext}_R(R/mR, R/nR) = R/dR$ where d is the highest common factor of m and n.

10. Let R be a Noetherian ring and M be a finitely generated R-module. Show that the following are equivalent for all R-modules N. (i) $\mathrm{Ext}^n_R(M, N) = 0$, (ii) $\mathrm{Ext}^n_{R_P}(M_P, N_P) = 0$

for every prime ideal P of R, and (iii) $\mathrm{Ext}^n_{R_Q}(M_Q, N_Q) = 0$ for every maximal ideal Q of R.

11. Let k be a field and let $R = k[X, Y]$. Let M be the trivial R-module $k[X, Y]/(X, Y)$. Use the Koszul complex to calculate $\mathrm{Ext}^n_R(M, M)$ for all $n \geq 0$.

12. Show that $\mathrm{Ext}_R(M, N)$ is independent of the choice of projective presentation for M.

13. Let K be a finite field extension of a field k. Show that it is a separable k-algebra exactly when it is a separable field extension of k.

14. Let K be the kernel of the k-linear map from $R \otimes_k R$ to R sending $r_1 \otimes r_2$ to $r_1 r_2$. Show that there is a derivation D from R to K such that the map from $\mathrm{Hom}_{R-R}(K, M)$ to $\mathrm{Der}(R, M)$ sending θ to the composition of D with θ is an isomorphism. Which θ correspond to inner derivations?

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