III Algebra   Michaelmas Term 2019

EXAMPLE SHEET 3

All rings are commutative with a 1 unless stated otherwise.

1. A chain of prime ideals is maximal if it is not a proper subset of another chain of primes. Prove that all maximal chains of prime ideals in a finitely generated k-algebra T which is an integral domain are of the same length, and that $\mathrm{ht}P + \mathrm{dim}T/P = \mathrm{dim}T$ for any prime ideal P of T.

2. Give an example of a finitely generated algebra T with a prime ideal P for which $\mathrm{ht}P + \mathrm{dim}T/P < \mathrm{dim}T$.

3. Let R be a Noetherian regular local ring. Show that $R[[X]]$ is a regular local ring of dimension $\mathrm{dim}R+1$. Deduce that if k is a field then $k[[X_1, \dots, X_n]]$ of formal power series in n indeterminates is a regular local ring of dimension n.

4. For a not necesarily commutative ring R show that the following are equivalent for a right ideal I: (i) $I \leq \mathrm{Jac}R$; (ii) if M is a finitely gebnerated R-module and a submodule N satisfies $N + MI = M$ then $N = M$; (iii) the set of elements $1 + x$ for $x \in I$ form a subgroup of the unit group of R.

5. For a not necessarily commutative ring R show that if an R-module M is a sum of simple submodules then M may be expressed as the direct sum of some simple submodules.

6. Let R be a semisimple right Artinian ring and M be an Artinian right R-module. Show that M is a direct sum of finitely many simple R-modules.

7. Let R be a k- algebra where k is an algebraically closed field, and suppose that R is finite dimensional as a k-vector space and semisimple. Define a Lie bracket on R by $[x, y] = xy - yx$. Show that the k vector space dimension of $R/[R, R]$ is equal to the number of isomorphism classes of simple right R-modules.

8. Let k be a field of characteristic $p > 0$ and let G be a finite group of order a power of p. Show that the augmentation ideal of kG (the kernel of the ring homomorphism from kG to k sends each g to 1) is nilpotent and that up to isomorphism the only simple module of kG is the trivial module, one dimensional as a k vector space.

9. Let $G = S_3$ and let k be a field of characteristic 2. Describe the simple modules, the socle series and the Jacobson radical of kG.

10. Show that a ring R with an exhaustive and separated filtration is an integral domain if the associated graded ring $\mathrm{gr}R$ is an integral domain. Assume that the filtration of R is

positive and show that R is Noetherian if gr$R$ is Noetherian. Is the same true for negative filtrations, for example the P-adic filtration of R where P is a prime ideal?

11. Let R be a Noetherian ring with ideal I. Show that the Rees ring of R with respect to the I-adic filtration is Noetherian. Let M be a finitely generated R-module. A filtration of M with respect to the I-adic filtration of R is said to be *good* if its Rees module $Rees(M)$ is a Noetherian $Rees(R)$-module. Show that this is equivalent to it being *stable* (i.e there is some J such that $M_{-J-j} = I^j M_{-J}$ for all $j > 0$).

12. (Artin, Rees) Let R be a Noetherian ring, and let I be an ideal. Let M be a finitely generated R-module with submodule N. Show that there exists $r \geq 0$ such that $N \cap I^a M = I^{a-r}(N \cap I^r M)$ for $a \geq r$.

13. (Krull) Let R be a Noetherian local ring, and let I be a proper ideal. Let M be a finitely generated R-module. Show that the intersection of all the submodules $I^n M$ is zero. In particular the intersection of all ideals I^n is zero.

14. Let R be a Noetherian ring and let I be an ideal. Let $S = 1 + I$. Show that the kernel of the canonical map from R to $S^{-1}R$ is the intersection of the positive powers of I.

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