III Algebra Michaelmas Term 2018

EXAMPLE SHEET 3

All rings are commutative with a 1 unless stated otherwise.

1. A chain of prime ideals is maximal if it is not a proper subset of another chain of primes. Prove that all maximal chains of prime ideals in a finitely generated k-algebra T which is an integral domain are of the same length, and that $\mathrm{ht}P + \mathrm{dim}T/P = \mathrm{dim}T$ for any prime ideal P of T.

2. Give an example of a finitely generated algebra T with a prime ideal P for which $\mathrm{ht}P + \mathrm{dim}T/P < \mathrm{dim}T$.

3. Let R be a Noetherian regular local ring. Show that $R[[X]]$ is a regular local ring of dimension $\mathrm{dim}R + 1$. Deduce that if k is a field then $k[[X_1, \ldots, X_n]]$ of formal power series in n indeterminates is a regular local ring of dimension n.

4. Let R be a k- algebra where k is an algebraically closed field, and suppose that R is finite dimensional as a k-vector space. Define a Lie bracket on R by $[x, y] = xy - yx$. Show that the k vector space dimension of $R/[R, R]$ is equal to the number of isomorphism classes of simple right R-modules.

5. Let k be a field of characteristic $p > 0$ and let G be a finite group of order a power of p. Show that the augmentation ideal of kG (the kernel of the ring homomorphism from kG to k sends each g to 1) is nilpotent and that up to isomorphism the only simple module of kG is the trivial module, one dimensional as a k vector space.

6. Let $G = S_3$ and let k be a field of characteristic 2. Describe the simple modules, the socle series and the Jacobson radical of kG.

7. Let R be a ring and let E be an R-module. Show that the following are equivalent. (1) E is injective; (2) If $\mu : E \longrightarrow M$ is a monomorphism then there exists $\beta : M \longrightarrow E$ such that $\beta\mu$ is the identity map; (3) E is a direct summand in every module which contains E as a submodule.

8. Le R be a ring. An R-module is said to be *divisible* if, for every e in E and every r in R which is not a zero-divisor, there exists e' in E such that $e = re'$. Show that an injective R-module is necessarily divisible.

9. Let R be a principal ideal domain. Show that an R-module is injective if and only if it is divisible.

10. Let R be the ring of integers. Show that any R-module may be embedded in an injective R-module. Let S be a ring and let M be an injective R-module. Show that $\mathrm{Hom}_R(S, M)$ is an injective S-module. Deduce that any S-module can be embedded in an injective S-module.

11. Let R be a ring and let I and J be ideals. Show that (a) $\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/IJ$, and (b) $\mathrm{Tor}_2(R/I, R/J) = \ker(I \otimes_R J \longrightarrow IJ)$

12. Let R be the ring of integers. Show that $\mathrm{Ext}_R(R/mR, R/nR) = R/dR$ where d is the highest common factor of m and n.

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