III Algebra Michaelmas Term 2018

EXAMPLE SHEET 2

All rings on this sheet are commutative with a 1.

1. Show that r lies in the Jacobson radical of R if and only if $1 - rs$ is a unit for all s in R.

2. Show that for a proper ideal I of a Noetherian ring R the condition that R/I has only one associated prime P is equivalent to the condition that if ab lies in I but a does not then some power b^n lies in I. Show that if these conditions hold then P is the radical of I.

3. A ring is Artinian if it satisfies the descending chain condition on ideals. Show that the nilradical of an Artinian ring is nilpotent.

4. Show that in an Artinian ring all the prime ideals are maximal and that there are only finitely many of them.

5. Show that every Artinian ring is Noetherian.

6. Show that a Noetherian ring of zero dimension is Artinian.

7. Prove that any field which is finitely generated as a ring is finite.

8. Let $R \leq T$ be rings with $T \setminus R$ closed under multiplication. Show that R is integrally closed in T.

9. Show that being integrally closed is a local property of integral domains.

10. A valuation ring is an integral domain R such that for any x in the field K of fractions of R, at least one of x or x^{-1} lies in R. Show that in a valuation ring any finitely generated ideal is principal.

11. Let R be a valuation subring of a field K. The group U of units of R is a subgroup of the multiplicative group $K^\times$ of K. Let $\Gamma = K^\times/U$. If α and β are represented by x and $y \in K$ define $\alpha \geq \beta$ to mean $xy^{-1} \in R$. Show that this defines a total ordering on Γ which is compatible with the group structure (i.e. $\alpha \geq \beta$ implies $\alpha\gamma \geq \beta\gamma$ for all $\gamma \in \Gamma$). (In other words Γ is a totally ordered Abelian group. It is called the value group of A.) Let $v : K^\times \longrightarrow \Gamma$ be the canonical homomorphism. Show that $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$.

12. Let R be an integral domain and K be its field of fractions. Show that the integral closure of R in K is the intersection of all the valuation subrings of K that contain R.

13. Let $R \leq T$ be rings with T generated by n elements as an R-module. Show that over every maximal ideal of R there lies at most n maximal ideals of T.

14. Let T be a finitely generated k-algebra, integral over an algebra R and let P be a prime ideal of R. Show that T has only finitely many primes lying over P.

15. Give an example of a Noetherian integral domain which has maximal ideals of different heights.

16. Let k be a field. Show that every k-subalgebra R of $k[X]$ is a finitely generated k-algebra and is of dimension 1 if $R \neq k$.

17. Let $Q_1, \ldots, Q_n$ be prime ideals of a ring R. Let I be an ideal and suppose it is contained in the union of these primes. Show that I is contained in some Q_i.

18. Let R be a Noetherian ring and $P_1 < P_2$ be prime ideals of R. Suppose there is some other prime Q lying strictly between P_1 and P_2, and show that there are infinitely many such Q.

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