III Algebra   Michaelmas Term 2018

EXAMPLE SHEET 1

All rings on this sheet are commutative with a 1.

1. Prove that the direct product of finitely many Noetherian rings is Noetherian.

2. Show that the set of prime ideals in a ring possesses a minimal member (with respect to inclusion).

3. By considering trailing coefficient ideals, prove that a ring R is Noetherian if and only if the power series ring $R[[X]]$ is Noetherian.

4. Let R be a Noetherian ring and θ be a ring homomorphism from R to R. Prove that if θ is surjective then it is also injective.

5. Let S be a multiplicatively closed subset of a ring R, and M be a finitely generated R-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

6. Let R be a ring. Suppose that for each prime ideal P the local ring R_P has no non-zero nilpotent element. Show that R has no non-zero nilpotent element. If each R_P is an integral domain, is R necessarily an integral domain?

7. Let $\phi : M \longrightarrow N$ be an R-module map. Show that the following are equivalent: (i) ϕ is injective; (ii) $\phi_P : M_P \longrightarrow N_P$ is injective for each prime ideal P; (iii) $\phi_Q : M_Q \longrightarrow N_Q$ is injective for each maximal ideal Q.

Prove the analogous result for surjective maps.

8. A multiplicatively closed subset S of a ring R is *saturated* when $xy \in S$ if and only if both x and y are in S. Prove that (i) S is saturated if and only if $R \setminus S$ is a union of prime ideals. (ii) If S is a multiplicatively closed subset of R, there is a unique smallest saturated multiplicatively closed subset S' containing S, and that S' is the complement in R of the union of the prime ideals which do not meet S. If $S = 1 + I$ for some ideal I, find S'.

9. (Kaplansky) Show that an integral domain is a unique factorisation domain if and only if all its non-zero prime ideals contain a non-zero principal prime ideal. Use this to show that if R is a principal ideal domain then $R[[X]]$ is a unique factorisation domain.

10. Let R be the ring of integers. Construct universal R -bilinear maps

$$(R/3R) \times (R/3R) \longrightarrow (R/3R)$$

$$(R/6R) \times (R/10R) \longrightarrow (R/2R)$$

and show that, if r and s are coprime integers, then any R-bilinear map on $(R/rR) \times (R/sR)$ is zero.

11. Prove that for R-modules M, N and L

$$M \otimes (N \otimes L) \cong (M \otimes N) \otimes L.$$

12. Show that there can be an element in a tensor product $M \otimes N$ which cannot be written as a single term $m \otimes n$ for any elements $m \in M$ and $n \in N$.

13. Show that the universality of $\otimes$ implies that $M \otimes N$ is spanned by the elements $m \otimes n$.

14. Let I be an ideal of a ring R. Show that $(R/I) \otimes M$ is isomorphic to M/IM.

15. Let R be a local ring, and M and N be finitely generated R-modules. Prove that if $M \otimes N = 0$ then $M = 0$ or $N = 0$.

16. Let $R = k[X]$, and I and J be the ideals of R generated by $X - \alpha$ and $X - \beta$ respectively. Show that $(R/I) \otimes_R (R/J)$ is a cyclic R-module and identify its annihilator. Show that $(R/I) \otimes_k (R/J)$ is a cyclic R-module when using the diagonal action and identify its annihilator.

17. Let $R = k[X_1, X_2, \ldots]$ be the polynomial ring with countably infinite indeterminates and I be the ideal generated by all the elements X_i^i. Show that R/I is not Noetherian and that its nilradical is not nilpotent.

18. Let I be an ideal contained in the Jacobson radical of R, and let M be an R-module and N be a finitely generated R-module. Let θ be an R-module map from M to N. Show that if the induced map from M/IM to N/IN is surjective then θ is surjective

19. Let I be an ideal of a ring R, and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

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