This Example Sheet involves questions that require significant extensions of the material covered in lectures. Most are optional; on the other hand, several are core results, which are chosen to illustrate the richness and flexibility of forcing and set-theoretic methods.

1 Absoluteness Results

(i) Show that the following properties are expressible by $\Sigma_1$-formulas:
   (a) $\alpha$ is not a cardinal;
   (b) $cf(\alpha) \leq \beta$;
   (c) $\alpha$ is singular;
   (d) $|x| \leq |y|$.

(ii) Show that the following properties are expressible by $\Pi_1$-formulas:
   (a) $\alpha$ is a cardinal;
   (b) $cf(\alpha)$ is regular;
   (c) $\alpha$ is weakly inaccessible (i.e. $\alpha \neq \lambda^+$ for all $\lambda$);
   (d) $y = P(x)$;
   (e) the formula $V_x = y$.

2 Well-founded Relations and Absoluteness of $\mathbb{P}$-names

An $L_\infty$-formula $\varphi(\bar{x})$ is downward absolute (respectively, upward absolute) if whenever $M$ is a transitive class, $(\forall \bar{x} \in M)(\varphi(\bar{x}) \rightarrow \varphi(\bar{x})^M)$ (respectively, $(\forall \bar{x} \in M)(\varphi(\bar{x})^M \rightarrow \varphi(\bar{x}))$). A class $\mathcal{M}$ is an inner model of ZF if $\mathcal{M}$ is transitive, $\text{Ord} \subseteq \mathcal{M}$, and for every axiom $\varphi$ of ZF, $\mathcal{M} \models \varphi$ (i.e. $ZF \vdash \varphi^\mathcal{M}$).

(i) Show that $\Sigma_1$-formulas are upward absolute and $\Pi_1$-formulas are downward absolute.

(ii) Prove that if $V \models (\kappa$ is an inaccessible cardinal), then $\mathcal{A} \models (\kappa$ is an inaccessible cardinal) whenever $\mathcal{A}$ is an inner model of ZF.

(iii) Let $R$ be a binary relation over $X$. Prove that $R$ is well-founded if and only if there is an order-preserving function $\rho : X \rightarrow \text{Ord} : xRy \rightarrow \rho(x) < \rho(y)$. Deduce that the property “$R$ is well-founded” is $\Delta_1$ and hence absolute for standard transitive models of ZF.

(iv) Formulating carefully any relevant results about absoluteness of definitions by transfinite recursion, deduce that the property “$\tau$ is a $\mathbb{P}$-name” is absolute for transitive models of ZF. [Hint. Recast the definition of $\mathbb{P}$-name as a definition by transfinite recursion over a well-founded relation.]

3 Models of Fragments of ZFC; Closure Properties

For a cardinal $\kappa$, let $H_\kappa$ be the family of sets hereditarily of cardinality less than $\kappa$.
(i) Prove if \( \kappa = cf(\kappa) > \aleph_0 \), then \( H_\kappa \models ZFC^- \), where \( ZFC^- \) is the theory whose axioms are those of \( ZFC \) without the Power Set axiom.

(ii) Deduce that no proof of the existence of \( \mathbb{R} \) avoids some non-trivial use of the Power Set axiom. [HINT. Consider \( H_\aleph_1 \).]

(iii) Assume \( \kappa = cf(\kappa) \geq \aleph_2 \). Let \( \mathcal{M} \) be a countable elementary submodel of \( (H_\kappa, \in) \).

(a) Suppose \( \varphi(v) \) is an \( L_\in \)-formula with free variable \( v \) (possibly with parameters from \( M \)) such that in any model of \( ZFC^- \) there is exactly one element that satisfies \( \varphi(v) \). Show that if \( a \in H_\kappa \) and \( H_\kappa \models \varphi[a] \), then \( a \in M \).

(b) The ordinals \( \omega \) and \( \omega_1 \) belong to \( M \).

(c) If \( \{a, A, B, f\} \subseteq M, a \in A \) and \( f : A \to B \) is a function (in \( V \)), then \( f(a) \in M \).

(d) If \( x \in M \) is countable (in \( V \)), then \( x \subseteq M \).

(e) For every \( \alpha \in \omega_1 \cup \{\omega_1\} \), \( \alpha \cap M \) is an ordinal.

(f) If \( A = \{A_\alpha : \alpha < \omega_1\} \in M \), then \( A_\alpha \in M \) for every \( \alpha \in \omega_1 \cap M \).

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4 Replacement Schema and Power Set Axiom in Generic Extensions

Suppose \( G \subseteq P \) is generic over \( \mathcal{M} \).

(i) Prove that for each \( L_\in \)-formula \( \varphi, \mathcal{M}[G] \models Replacement_{\varphi} \). [HINT. Find a bounding ordinal \( \delta \in M \) such that \( V_\delta^\mathcal{M} \) contains a set of \( P \)-names whose values are sufficient to provide all the required elements in \( M[G] \).]

(ii) Prove that \( \mathcal{M}[G] \models PowerSet \). [HINT. For \( y \subseteq x, y \in M[G] \), let \( \tau = \{(r, \rho) \in P \times range(\check{x}) : r \vdash (\rho \in y)\} \); show \( \tau[G] = y \); let \( \check{u} = \{(p, \sigma) : \sigma \in M, \sigma \subseteq P \times range(\check{x})\} \).

Check that \( P(x)^{M[G]} \subseteq u \) and explain why this suffices.

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5 Concerning Forcings, Anti-chains and Generic Sets

Suppose that \( \mathbb{P} \) is a forcing in a model \( \mathcal{M} \) of \( ZFC \).

(i) Prove that a set \( G \subseteq P \) is generic over \( \mathcal{M} \) if and only if for every maximal anti-chain \( A \in M \) of \( \mathbb{P} \), \( G \cap A \models 1 \). [HINT. One direction uses AC.]

(ii) Assume that the forcing \( \mathbb{P} \) has a least element \( 0_\mathbb{P} \). A set \( D \subseteq P \) is:

(a) pre-dense above \( p \in P \) if \( (\forall q \in P)(q \geq p \to (\exists d \in D)(d \text{ and } q \text{ are compatible}); \)

D \text{ is pre-dense above } 0_\mathbb{M}:

(b) dense above \( p \in P \) if \( (\forall q \in P)(q \geq p \rightarrow (\exists d \in D)(d \geq q)) \). So \( D \) is dense in \( \mathbb{P} \)

if \( D \) is dense above \( 0_\mathbb{M} \).

Suppose that \( E \) is pre-dense in \( \mathbb{P} \) and \( G \subseteq P \) is generic over \( \mathcal{M} \). Show that \( G \cap E \neq \emptyset \).

Suppose that \( E \) is pre-dense above \( q \in \mathbb{P} \) and \( G \subseteq P \) is generic over \( \mathcal{M} \). Show that if \( q \in G \), then \( G \cap E \neq \emptyset \).

(iii) Deduce that the following are equivalent for a directed downward closed set \( G \subseteq P, P \in \mathcal{M} \) where \( \mathcal{M} \) is a transitive model of \( ZFC \).

(a) \( G \) is generic in \( \mathbb{P} \) over \( \mathcal{M} \);

(b) \( G \cap D \neq \emptyset \) for every dense open set \( D \subseteq P \) in \( \mathcal{M} \);

(c) \( G \cap C \neq \emptyset \) for every dense subset \( C \subseteq P \) in \( \mathcal{M} \);

(d) \( G \cap B \neq \emptyset \) for every pre-dense subset \( B \subseteq P \) in \( \mathcal{M} \);

(e) \( G \cap A \neq \emptyset \) for every maximal anti-chain \( A \subseteq P \) in \( \mathcal{M} \).
6 Optional. Forcing, Chain Conditions, and Elementary Submodels

For a forcing \( P \), a cardinal \( \kappa \) is large enough (for \( P \)) if \( \kappa = cf(\kappa) > \aleph_1 \) and the set of dense subsets of \( P \) belongs to \( H_\kappa \) (so in particular, \( P \), the conditions in \( P \) and every dense subset of \( P \) all belong to \( H_\kappa \)). For a set \( N \), a condition \( p \in P \) is called \( N \)-generic if for every \( D \in N \) which is a dense subset of \( P \), \( D \cap N \) is pre-dense above \( p \).

Suppose that \( \kappa \) is large enough for \( P \). Prove the following are equivalent:

(i) \( P \) has the countable chain condition;
(ii) for every countable elementary submodel \( N \) of \( H_\kappa \), \( 0_p \) is \( N \)-generic;
(iii) every countable subset \( X \) of \( H_\kappa \) is contained in an elementary submodel \( N \) of \( H_\kappa \) such that \( 0_p \) is \( N \)-generic.

[HINT. For (i) \( \Rightarrow \) (ii), consider an \( A \in N \) maximal relative to the property of being an anti-chain contained in \( D \). For (iii) \( \Rightarrow \) (i), show if \( A \in N \) is a maximal anti-chain, then \( A = \{ p \in P : (\exists q \in A)(q \leq_p p)\} \) is dense.]

7 The Forcing Relation

Suppose that \( P \) is a non-trivial forcing, \( p, q \in P \), and \( \varphi \) is a formula in the vocabulary of \( ZFC \) which may contain \( P \)-names. Show:

(i) if \( p \Vdash_P \varphi \) and \( p \leq_P q \), then \( q \Vdash_P \varphi \);
(ii) if \( q \Vdash_P \varphi \) for every \( q \geq_P p \) such that \( p \neq q \), then \( p \Vdash_P \varphi \);
(iii) if \((\forall \tau \in M^P)((p \leq_P \tau) \land \tau \Vdash_P \neg \varphi)\), then \( p \Vdash_P \neg \varphi \);
(iv) \((\exists \tau)(p \leq_P \tau)(\tau \text{ decides } \varphi)\), i.e. either \( r \Vdash_P \varphi \) or \( r \Vdash_P \neg \varphi \).
(v) if \( p \) does not decide \( \varphi \), then \( \bigwedge_{i=1,2}(\exists \tau_i)(p \leq_P \tau_i)(\tau_1 \Vdash_P \varphi) \land (\tau_2 \Vdash_P \neg \varphi) \).

8 Names

Suppose \( G \subseteq P \) is generic over \( M \).

(i) Suppose \( \sigma, \tau \in M^P \). Show \( \sigma \cap G \cup \tau \cap G = (\sigma \cup \tau)_G \).
(ii) Suppose \( \tau \in M^P \) and \( \text{range}(\tau) \subseteq \{ \bar{n} : n \in \omega \} \). Let \( \sigma = \{ \bar{n} : (\forall q \in P)((q, \bar{n}) \in \tau \rightarrow p \perp q)\} \). Show that \( \sigma \cap G = \omega \setminus \tau \cap G \). [HINT. The set \( \{ r \in P : (\exists p \leq r)((p, \bar{n}) \in \sigma \land (p, \bar{n}) \in \tau)\} \) is dense.]
(iii) Suppose \( A \) is an anti-chain in \( P \) and for each \( a \in A \), \( \tau_a \) is a \( P \)-name. Show there exists a \( P \)-name \( \tau \) such that for every \( a \in A \), if \( a \in G \), then \( \tau[G] = \tau_a[G] \), and \( \tau[G] = \emptyset \) if \( G \cap A = \emptyset \). [HINT. Suppose \( \tau_a = \{ (q_{a,j}, \tau_{a,j}) : j < i_a \} \). Consider the \( P \)-name \( \tau = \{ (\tau_{a,j}) : a \in A, j < i_a, r \geq q_{a,j}, \text{ and } r \geq a \} \). This provides a useful way of constructing names from other names indexed by elements of anti-chains.]
9 Nice Names and Bounds for the Continuum

Suppose $P \in M$ and $G \subseteq P$ is generic over $M$. A name $\tau \in M^P$ is a nice name for a subset $x$ of $\sigma \in M^P$ if $\tau = \bigcup \{A_\pi \times \{\pi \in \text{range}(\sigma)\} : A_\pi$ is an anti-chain in $P$. 

(i) Prove that for all $\sigma, \rho \in M^P$ there exists a nice name $\tau$ such that $\Vdash_P (\rho \subseteq \sigma \rightarrow \rho = \tau)$. [Hint. For $\pi \in \text{range}(\sigma)$, let $A_\pi$ be maximal relative to the properties (1) $(\forall p \in A_\pi)(p \Vdash_P \pi \in \rho)$ and (2) $A_\pi$ is an anti-chain in $P$; refer to a previous Question to check that $\tau$ as defined works.]

(ii) Suppose $(P$ is a c.c.c. forcing and $\lambda$ is a cardinal $)^M$. Let $\kappa^s = (|P|^\lambda)^M$. Then $(2^\lambda \leq \kappa^s)^{M[G]}$. [Hint. Count the number of nice names for the members of $P(\lambda)^{M[G]}$, remembering that $P$ has the countable chain condition.]

(iii) Deduce that if $(\lambda$ is a cardinal and $\lambda^{\aleph_0} = \lambda)^M$, then there is a generic extension $M[H]$ such that $(2^{\aleph_0} = \lambda)^{M[H]}$.

10 Forcing and the Independence of Diamond

(i) Prove that the theory $ZFC + \diamondsuit$ is relatively consistent. [Hint. It may be easier to verify $\diamondsuit$ in its functional form (see Example Sheet 3). Let $I = \{\langle \alpha, \zeta \rangle : \zeta < \alpha < \omega_1^M\}$ and consider the forcing $Q = Fn(I, 2, \aleph_1^M)$. Show that $Q$ is countably complete, and that if $G$ is generic over $M$, then in $M[G]$, a $\diamondsuit$-sequence is provided by $(\bigcup \langle G\rangle_\alpha : \alpha < \omega_1)$. For this, noticing that $Q$ adds no new $\omega$-sequences and $\aleph_1^M = \aleph_1^{M[G]}$, define a sequence of ordinals and conditions forcing an arbitrary club to intersect the family of guesses for a function $f : \omega_1 \rightarrow \omega_1$. (Refer to Kunen, Set Theory, chapter VII, or Shelah, Proper and Improper Forcing, chapter 1, if difficulties arise. Remark: $\diamondsuit$ is true in $L$ (as was explained in the talk by Professor Mathias); this was the earliest proof of its consistency, due to Jensen.)

(ii) Deduce that $\diamondsuit$ is independent of $ZFC$.

(iii) Show if $(\lambda$ is a cardinal and $\lambda^{\aleph_0} = \lambda$ and $\diamondsuit)^M$, then there is a generic extension $M[H]$ such that $M[H] \models (\diamondsuit^{\aleph_0} = \lambda$ and there is a Suslin tree).

(iv) Suppose that $M \models (\langle P$ is a c.c.c. forcing and $|P| \leq \aleph_1$ and $\diamondsuit \rangle$. Prove that for every $G \subseteq P$ generic over $M$, $M[G] \models \diamondsuit$. [Hint. In $M$, use $\diamondsuit$ to guess nice names for subsets of $\omega_1$.]

(v) Deduce that $\diamondsuit$ does not imply $V = L$.

(vi) Suppose that $M \models (\langle P$ is a c.c.c. forcing and $|M[G]| \models \diamondsuit$, where $G \subseteq P$ is generic over $M$. Show that $M \models \diamondsuit$. [Hint. Recall the equivalent characterisations of $\diamondsuit$ from Example Sheet 3 and the lemma about approximating functions in c.c.c. generic extensions.]

(vii) Optional. Prove that $\clubsuit$ is independent of $ZFC$.

11 Adding Cohen Reals and Suslin Trees

(i) A tree $T$ is ever-branching if for every $s \in T$, the set $\{t \in T : s \leq_T t\}$ is not linearly ordered. Let $M$ be a CTM such that $(T$ is an ever-branching Suslin tree) $^M$. Suppose that $(P = Fn(\lambda \times \omega, 2, \aleph_0) \land \lambda \geq \aleph_0)^M$. Prove that for any set $G \subseteq P$ generic over $M$, $M[G] \models (T$ is a Suslin tree).
(ii) Deduce that there is a model of \( ZFC \) in which there is a Suslin tree but \( CH \) fails.

Remark: So the existence of Suslin trees does not imply \( CH \) (nor \textit{a fortiori} \( \Diamond \)). The proof that \( CH \) does not imply Suslin trees is due to Jensen.


Suppose that \( \mathbb{P} \) is a separative countable forcing and \( \mathbb{M} \) is a CTM. Prove Shelah’s theorem: in every \( \mathbb{P} \)-generic extension of \( \mathbb{M} \), there is a Suslin tree. In particular, the Cohen forcing adds a Suslin tree.

Remark: This result provides a direct proof by forcing of the relative consistency of the failure of Suslin’s Hypothesis. Adding Cohen reals has \( \Diamond \)–like combinatorial consequences; it is far from innocuous.

13 The Levy Collapse

The Levy collapse is the forcing \( \text{Lv}(\kappa) \) defined as follows:

\[
\{ p : p \text{ is a finite function, } \text{dom}(p) \subseteq \kappa \times \omega, \text{ and } (\forall (\alpha,n) \in \text{dom}(p))(p(\alpha,n) \in \alpha) \} 
\]

with the partial ordering \( p \leq q \) iff \( p \subseteq q \).

Suppose that \( \kappa \) is strongly inaccessible in the countable transitive model \( \mathbb{M} \).

(i) What is the cardinality of a maximal anti-chain in \( \text{Lv}(\kappa) \)?

(ii) What is the largest cardinal \( \lambda \) for which \( \text{Lv}(\kappa) \) is \( \lambda \)-closed?

(iii) Which cardinals are preserved and which are collapsed under forcing with \( \text{Lv}(\kappa) \)?

(iv) Let \( H \) be generic in \( \text{Lv}(\kappa) \) over \( \mathbb{M} \). Prove that \( \kappa = \aleph_{\mathbb{M}[H]} \).

(v) Suppose \( \kappa \) is strongly inaccessible in the CTM \( \mathbb{M} \) and let \( H \) be generic in \( \text{Lv}(\kappa) \) over \( \mathbb{M} \). Prove that \( KH \) holds in the generic extension \( \mathbb{M}[H] \). [ Hint. Use the previous part and refer to Example Sheet 3.] Comments: \( \text{Con}(ZFC + KH) \) can be proved from \( \text{Con}(ZFC) \) but a more complicated forcing is involved (see Kunen, chapter VII, Exercise H19). However, \( \text{Con}(ZFC + \neg KH) \), which requires an iterated forcing argument, \textit{does} require the hypothesis of the existence of a strongly inaccessible cardinal, because \( \neg KH \) implies \( \aleph_2 \) is inaccessible in \( L \) (see Kunen, Chapter VII, Exercise B9).

14 Zermelo, Strong Inaccessibility and Second-Order Set Theory

(i) Suppose \( \kappa = cf(\kappa) > \aleph_0 \). Prove that \( \kappa \) is a strongly inaccessible cardinal if and only if \( V_\kappa = H_\kappa \).

(ii) Suppose \( \kappa = cf(\kappa) > \aleph_0 \). Show that if \( (H_\kappa, \in) \) is a model of \( ZFC \), then \( \kappa \) is a strongly inaccessible cardinal.

(iii) Let \( ZFC^2 \) denote the second-order axiomatic system obtained from \( ZFC \) in which the Replacement Schema is expressed as the following single axiom with a universal second-order quantifier: \( (\forall C)(\forall z)(\forall u)(\forall v)((z,v) \in C \wedge (z,u) \in C \Rightarrow v = u) \Rightarrow \exists x \forall y(y \in x \Leftrightarrow \exists z(z \in u \wedge (z,y) \in C)) \), i.e. if \( C \) is a functional class and \( a \) is a set, then \( \{ C(z) : z \in a \} \), the image of \( a \) under \( C \), is also a set. The intended interpretation of the second-order variables ranges over arbitrary subsets of the domain. Notice that models of \( ZFC^2 \) are well-founded.

Show that if \( \kappa \) is a cardinal such that \( V_\kappa \models ZFC^2 \), then \( \kappa \) is strongly inaccessible.
(iv) For each strongly inaccessible cardinal $\kappa$, there is up to isomorphism exactly one model of $ZFC^2$ with the set of ordinals of order type $\kappa$, namely, $(V_\kappa, \in)$. [Hint. Mostowski Collapse; induction on rank.]

15 The Generalized $\Delta$-System Lemma

(i) Suppose that $\lambda < \kappa = cf(\kappa)$ and $\bigwedge_{\alpha < \kappa} \alpha^\lambda < \kappa$. Prove that if $|A| = \kappa$ and $x \in A$ implies $|x| < \lambda$, then there exists $B \subseteq A$ such that $|B| = \kappa$ and $(\exists \gamma)(\forall x \in B)(x \not\rightarrow x \cap y = r)$. [Hint. This is a standard result. WLOG, $\bigcup A \subseteq \kappa$ and some fixed $\gamma < \lambda$ is the order type of every $x = \{x(\xi) : \xi < \rho\} \in A$. Using $\bigwedge_{\alpha < \kappa} \alpha^\lambda < \kappa$ and $\kappa = cf(\kappa)$, let $\xi_0$ be the minimal $\xi$ such that $\{x(\xi) : \xi \in A\}$ is cofinal in $\kappa$; let $\sigma = \sup\{x(\eta) + 1 : x \in A \land \eta < \xi_0\}$, so (*) $x \mid \xi_0 \subseteq \sigma < \kappa$; now define by induction $\{x(\alpha) : \alpha < \kappa\}$ such that $x_\alpha(\xi_0) > \max\{\sigma, \sup\{x_\beta(\eta) : \beta < \alpha \land \eta < \rho\}\}$. Use (*) and $\sigma^\lambda < \kappa$ to refine $\{x(\alpha) : \alpha < \kappa\}$ and extract a root $r \subseteq \sigma$.]

(ii) Find a family of $\aleph_\omega$ finite sets such that no subfamily of size $\aleph_\omega$ has a root.

16 Optional. Moderately Large Cardinals Do Not Decide $CH$

A cardinal $\kappa$ is called Ramsey if $\kappa \rightarrow (\kappa)^2_2$, i.e., every colouring $f : [\kappa]^{<\omega} \rightarrow 2$ of all the finite subsets of $\kappa$ by 2 colours has a monochromatic subset of cardinality $\kappa$. In 1961, Scott proved that the existence of large enough cardinals contradicts the axiom $V = L$. However, the Continuum Hypothesis appears to behave differently.

(i) Show that the set $\lambda^2$ with the lexicographic order $\preceq_{\text{lex}}$ contains no increasing or decreasing sequences of length $\lambda^+$. [Hint. Otherwise suppose $H$ is e.g. $\preceq_{\text{lex}}$-increasing of size $\lambda^+$; WLOG, $H = \{h_\alpha : \alpha < \lambda^+\}$ and for some least $\gamma \leq \lambda, \forall g, h \in H, g \upharpoonright \gamma \neq h \upharpoonright \gamma$. Now find $\xi^* < \gamma$ such that $\{h_\alpha \mid \xi^* < \alpha < \lambda^+\}$ has cardinality $\lambda^+$.]

(ii) Prove that $2^\lambda \rightarrow (\lambda^+)^2_2$. [Hint. Otherwise, consider a homogeneous set $Y$ of size $\lambda^+$ for the 2-colouring $F$ of $[\lambda^+]^2$ given by $F(\{f, g\}) = 0$ if and only if $f \preceq_{\text{lex}} g$. This is due to Sierpiński and Kurépa independently.]

(iii) Prove that if $\kappa$ is a Ramsey cardinal, then $\kappa$ is strongly inaccessible. [Hint. Regularity is easy: use the colouring $c(\{(\alpha, \beta) \mid \alpha, \beta \in X_\xi\}) = 0$ where $\kappa = \bigcup_{\xi < \gamma} X_\xi, |X_\xi| < \kappa$; for strong inaccessibility, use the previous part.]

(iv) Prove that $\kappa$ is a Ramsey cardinal if and only if for all $\beta < \kappa, \kappa \rightarrow (\beta)^2_2$. [Hint. For the hard direction, if $f : [\kappa]^{<\omega} \rightarrow \beta$, define a 2-colouring $g$ as follows: $g(\{\xi_1, \ldots, \xi_\eta\}) = 0 \iff n = 2m \land f(\{\xi_1, \ldots, \xi_m\}) = f(\{\xi_{m+1}, \ldots, \xi_{2m}\})$. Notice if $H$ is homogeneous for $g$ and of cardinality $\kappa$, then $g \upharpoonright [H]^n \equiv 0$ and so $H$ is homogeneous for $f$ also.]

(v) Suppose that $M \models (\kappa$ is a Ramsey cardinal and $P$ is a forcing of cardinality less than $\kappa$). Let $G$ be $P$-generic over $M$. Prove that $M[G] \models (\kappa$ is a Ramsey cardinal). [Hint. If $M[G] \models (\tau_\kappa$ is a colouring of $[\kappa]^{<\omega}$), and this statement is forced by some condition $p \in G$, consider the colouring $g : [\kappa]^{<\omega} \rightarrow P(\mathbb{P} \times 2)$ defined by $g(\alpha) = \{\langle q, x \rangle : p \leq q \land q \Vdash (\tau(\bar{a}) = \bar{x})\}$. Note that $g \in M$ and $P(\mathbb{P} \times 2)$ has cardinality $\beta$ for some $\beta < \kappa$ (by strong inaccessibility). Use the previous part, in $M$, to obtain a homogeneous set $Y \in M$; show $p \Vdash (Y$ is homogeneous for $\tau)$.]

(vi) Deduce that $CH$ is independent of $ZFC + (\exists \kappa)(\kappa$ is a Ramsey cardinal).
REMARK. This type of result in very general form is due to Levy and Solovay (1967); see A. Kanamori, *The Higher Infinite*, Springer, 2009. The power of large cardinals to decide a statement ϕ is thus circumscribed by the existence of forcings of relatively small size, if such forcings can be used to establish the relative independence of ϕ from ZFC.

17 Optional. Embeddings and Elementary Embeddings of V

An uncountable cardinal κ is measurable if there exists a κ-complete non-principal ultrafilter U over κ. Suppose that M is a (transitive) model of ZFC. While the product structure Mκ does not satisfy much of ZFC, the ultrapower Mκ/U of M relative to U is a model of ZFC. Its elements are the equivalence classes (f)U of the relation ≡U defined by g ≡U h if {α < κ : g(α) = h(α)} ∈ U for g, h ∈ Mκ. (Care is needed in the case where M is a proper class; one uses the restricted equivalence classes consisting of elements of least rank, a stratagem called Scott’s trick.) The mapping ωU : M → Mκ/U defined by ωU(a) = (a*)U where a*(α) = a for all α < κ is an elementary embedding, and so Mκ/U is a model of ZFC. If U is R₂-complete, then Mκ/U is also well-founded and satisfies the hypotheses of Mostowski’s Lemma. Its transitive collapse under the Mostowski collapsing map π is denoted M_U. Taking M = V, the associated mapping j_U : V → V_U, defined by j_U(a) = π(ωU(a*)) is an elementary embedding of V into V_U.

(i) **Scott’s Theorem; 1961**

Show that if there is a measurable cardinal, then V \neq L. [**HINT.** Suppose that κ is a measurable cardinal; what is the first ordinal moved by j_U? If κ is the smallest measurable cardinal, what will it be in the ultrapower?]

(ii) **Kunen’s Theorem; 1971**

Prove that no non-trivial elementary embedding exists from V into V. [**HINT.** Otherwise, use question 7(iv) from Example Sheet 3 to derive a contradiction; for details, see A. Kanamori, *The Higher Infinite*, Springer, 2009, pages 319–320.]

REMARK. Joel Hamkins has shown very recently that every countable model of set theory (M, ∈_M), including every well-founded model, is isomorphic to a submodel of its own constructible universe (L_M, ∈_L). In terms of embeddings (i.e. injective homomorphisms), there is an embedding j : (M, ∈_M) → (L_M, ∈_L) that is elementary for quantifier-free assertions in the language of set theory. See: Hamkins, J.D., J. Math. Log., 13, 1350006 (2013) [27 pages].

18 Optional. Forcing and Partial Isomorphisms

Suppose that A and B are τ-structures in a vocabulary τ. Say that A and B are partially isomorphic, denoted A \simeq_p B if some non-empty family F \subseteq Part(A, B) of the partial isomorphisms from A to B is a back-and-forth set for A and B:

\[(\forall f \in F)(\forall g \in F)((f \subseteq g \land a \in dom(g)) \text{ and } (\forall f \in F)(\forall b \in B)((g \subseteq f \land b \in range(g))).\]

(i) Prove that if \(\tau, A, B\) are countable and \(A \simeq_p B\), then \(A \simeq B\).

(ii) Show that the converse of (i) fails. [**HINT.** Consider the linear orders \(\mathbb{Q}\) and \(\mathbb{R}\).]

(iii) Prove that if two structures A and B are partially isomorphic, then there is a forcing extension in which they are isomorphic.

REMARK. Partial isomorphism yields a characterization of elementary equivalence in the infinitary language L_{\omega_1, \omega}. For a recent introduction to these ideas, see J. Väänänen, *Models and Games*, Cambridge University Press, 2011.
19 Optional. Martin’s Maximum

A forcing \( P \) is called stationary-preserving if \( P \) does not destroy stationary subsets of \( \omega_1 \): if \( M \models (S \text{ is a stationary subset of } \omega_1) \), then \( M[G] \models (S \text{ is a stationary subset of } \omega_1) \), whenever \( G \) is \( P \)-generic over \( M \). Martin’s Maximum is the statement \( MM \): for every stationary-preserving forcing \( P \), if \( (\forall \alpha < \omega_1)(D_\alpha \text{ is dense open in } P) \), then there exists a \( \{D_\alpha : \alpha < \omega_1\} \)-generic set \( G \) in \( P \).

(i) Show that if \( P \) has the countable chain condition, then \( P \) is stationary-preserving.
(ii) Give an example of a stationary-preserving forcing that has an uncountable anti-chain.
(iii) Prove \( MM \) implies \( MA_{\aleph_1} \): for every c.c.c. forcing \( P \), if \( \{D_\alpha : \alpha < \omega_1\} \) is a family of dense sets in \( P \), then there exists a \( \{D_\alpha : \alpha < \omega_1\} \)-generic set \( G \) in \( P \).
(iv) Show \( MA_{\aleph_1} \) implies Suslin’s Hypothesis.

REMARK. The relative consistency strength of \( ZFC + MM \) is far stronger than that of \( ZFC + MA \) which is equiconsistent with \( ZFC \). \( MM \) requires a large cardinal axiom for its consistency.

20 Optional. Normal Functions and Mahlo Cardinals

A (class) function \( G : \text{Ord} \to \text{Ord} \) is called normal if \( G \) is (strictly) increasing \( (\alpha < \beta \rightarrow G(\alpha) < G(\beta)) \) and continuous (for all limit \( \delta \in \text{Ord}, G(\delta) = \bigcup_{\alpha < \delta} G(\alpha) \) ). A (strongly) inaccessible cardinal \( \kappa \) is called (strongly) Mahlo if \( \{\alpha < \kappa : \alpha = \text{cf}(\alpha)\} \) is stationary in \( \kappa \).

(i) (a) Prove in \( ZFC \) that every normal function \( G \) has a fixed point: there is \( \delta \in \text{Ord} \) such that \( G(\delta) = \delta \).

(b) Call the statement “every normal function has a regular fixed point” the regular fixed point axiom \( RFPA \). Show that \( RFPA \) is not provable in \( ZFC \).

(ii) (a) Suppose \( \kappa \) is strongly Mahlo. Show \( \{\alpha < \kappa : \alpha \text{ is strongly inaccessible}\} \) is stationary in \( \kappa \)

(b) Prove that if \( \kappa \) is strongly Mahlo, then \( V_\kappa \models RFPA \).

21 The Axiom of Choice in \( L \)

(i) Define by transfinite recursion a \( \Delta_1 \) well-ordering \( <_\alpha \) of \( L_\alpha \) for \( \alpha \in \text{Ord} \) and a \( \Sigma_1 \) well-ordering \( <_L \) of \( L \). [HINT. The case \( <_\delta \) for limit \( \delta \) is immediate (take unions). For the successor case \( <_{\beta+1} \), recall that the codes of formulas without parameters can be identified with elements of \( L_\omega \) and can be well-ordered using the lexicographic order; since the parameters of a set in \( L_{\beta+1} \) arise already in \( L_\beta \), they can be well-ordered by \( <_\beta \). So the predicate \( x <_{\beta+1} y \iff (x \in L_\beta \land y \in L_\beta \land x <_\beta y) \lor (x \in L_\beta \land y \in (L_{\beta+1} \setminus L_\beta)) \lor (x \in (L_{\beta+1} \setminus L_\beta) \land y \in (L_{\beta+1} \setminus L_\beta) \land \text{the first formula defining } x \text{ over } L_\beta \text{ precedes the first formula defining } y \text{ over } L_\beta \). For the global case, note that the predicate \( x <_L y \iff (\exists \alpha)(x \in L_\alpha \land x <_\alpha y) \) has the required complexity.]

(ii) Deduce that \( V = L \) implies \( AC \).

(iii) Conclude that \( L \) is a model of \( ZFC \).
(i) **Prove Dilworth’s theorem:** If a partial order $\mathbb{P}$ has at least $n^2 + 1$ elements for some $n < \omega$, then it has either a chain of size $n + 1$ or an anti-chain of size $n + 1$. Equivalently, if the largest anti-chain in $\mathbb{P}$ has size $n$, then $\mathbb{P}$ can be decomposed into $n$ chains.

(ii) **Suggest and prove a generalization of Dilworth’s theorem for infinite cardinals.** Give examples to support the alleged optimality of your result.

(iii) **Galvin’s Conjecture**

Let $\kappa_D$ be the least cardinal $\kappa$, if it exists, such that for every partial order $\mathbb{P}$, if every suborder of $\mathbb{P}$ of size less than $\kappa$ can be decomposed into countably many chains, then $\mathbb{P}$ can also be decomposed into countably many chains. Can you eliminate $\aleph_0$ and $\aleph_1$ as candidate values for $\kappa_D$? **Galvin’s Conjecture** states that $\aleph_2$ is a possible value for $\kappa_D$. See S. Todorcevic, *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1-72.

**REMARK.** After the classic works of Gödel and Cohen, the following are accessible and list many further suggestions for reading and research.


Dehornoy, P., *Recent progress on the Continuum Hypothesis (after Woodin)*;


http://www.logic.harvard.edu/EFI_CH.pdf


http://www.math.yorku.ca/~steprans/Research/PDFSOfArticles/hoc2INDEXED.pdf

**REMARK.** For research problems in set theory, go to the sources; there are some treasure houses.


http://shelah.logic.at/files/702.pdf

Fremlin, D.H., *Problems*;

https://www.essex.ac.uk/maths/people/fremlin/problems.pdf

Miller, A.W., *Some interesting problems*;
