1 Clubs

(i) Suppose $\kappa = cf(\kappa) > \aleph_0$. Show $U = \{ \alpha < \kappa : \omega \alpha = \alpha \}$ and $V = \{ \alpha < \kappa : 2^\alpha = \alpha \}$ are club in $\kappa$. Give an example of two disjoint clubs in $\aleph_\omega$. Do there exist disjoint stationary subsets of $\aleph_1$ and of $\aleph_2$?

(ii) Suppose that $A$ is club in $\kappa = cf(\kappa) > \aleph_0$ and $f : A \subseteq \kappa \to \kappa$ is a function. Prove that $B = \{ \alpha \in A : (\forall \xi < \alpha)(f(\xi) < \alpha) \}$ is club in $\kappa$. Deduce that if $\lambda < cf(\kappa)$ and $F$ is a family of $\lambda$ many functions from $A$ into $\kappa$, then the set $C = \{ \alpha \in A : (\forall f \in F)(\forall \xi < \alpha)(f(\xi) < \alpha) \}$ contains a club in $\kappa$. In other words, the ordinals which are closed under members of $F$ contain a club.

(iii) Suppose $D \subseteq \kappa = cf(\kappa) > \aleph_0$. Show that $D$ is a club of $\kappa$ if and only if $D$ is the range of a continuous strictly increasing function $f : \kappa \to \kappa$.

(iv) Prove that if $\delta > cf(\delta) > \aleph_0$, then there is a club $E$ of $\delta$ such that no member of $E$ is a regular cardinal. [Hint. Try the range of a continuous function $f : \kappa \to \{ \alpha \in \kappa : \alpha > cf(\kappa) \}$.

(v) Optional. Suppose $\mathbb{M}$ is a $\tau$-structure with domain $\kappa$ where $\tau$ is a vocabulary of cardinality less than $\kappa = cf(\kappa) > \aleph_0$. Show that $\{ \delta \in \kappa : \mathbb{M} \models \delta \}$ is an elementary substructure of $\mathbb{M}$. [Hint. This assumes some first-order model theory: use Skolem functions and/or the elementary chain theorem.]

2 Non-stationary sets

Suppose $\kappa = cf(\kappa) > \aleph_0$, and $X_\alpha$ is non-stationary in $\kappa$ for $\alpha < \kappa$.

(i) Show $\bigcup_{\alpha < \kappa}(X_\alpha \setminus (\alpha + 1))$ is non-stationary in $\kappa$.

(ii) If $\{ X_\alpha : \alpha < \kappa \}$ is pairwise disjoint, prove $X = \bigcup_{\alpha < \kappa}X_\alpha$ is stationary if and only if $B = \{ \min(X_\alpha) : \alpha < \kappa \}$ is stationary.

(iii) If $\{ X_\alpha : \alpha < \kappa \}$ is pairwise disjoint, then there exists $a \in [\kappa]^\omega$ such that $\bigcup_{\alpha \in a}X_\alpha$ is non-stationary.

3 Applications

(i) Suppose $f : \omega_1 \to \mathbb{R}$ is a continuous function, where $\omega_1$ has the order topology. Prove $(\exists \alpha < \omega_1)(\forall \beta > \alpha)(f(\beta) = f(\alpha))$, i.e. $f$ is eventually constant.

(ii) The ordinal $\omega_1$ with the order topology is not a metrizable topological space.

(iii) Prove the following result of Shelah. Suppose $S$ is a stationary subset of $\kappa = cf(\kappa) > \aleph_0$ and $g$ and $h$ are functions from $S$ into $\lambda$ such that $(\forall \xi \in S)(g(\xi) \neq h(\xi))$. Then there exists a stationary subset $S' \subseteq S$ such that:

$$\{ g(\xi) : \xi \in S' \} \cap \{ h(\zeta) : \zeta \in S' \} = \emptyset.$$
Non-stationary Ideals

Splitting Stationary Sets and Solovay’s Theorem

Club Filters

(iii) Suppose \( S \subseteq S \) such that \( (\forall \zeta, \eta \in D_1)((g(\zeta) < \zeta \Leftrightarrow g(\eta) < \eta) \land (h(\zeta) < \zeta \Leftrightarrow h(\eta) < \eta)) \); apply Fodor’s Lemma to find a stationary subset \( D_2 \subseteq D_1 \) such that if \( g(\zeta) < \zeta \) for all \( \zeta \in D_1 \), then \( g \) is constant on \( D_2 \) and the same for \( h \); now use Question 1 to obtain a club \( C \) closed under the functions \( g \) and \( h \). Let \( S' = D_2 \cap C. \)

4 Club Filters

A filter over a non-empty set \( I \) (or on \( P(I) \), in \( P(I) \), or sometimes on \( I \)) is a family \( F \subseteq P(I) \) such that

(i) \( \emptyset \notin F, I \in F \);
(ii) if \( X, Y \in F \), then \( X \cap Y \in F \);
(iii) if \( X \in F, X \subseteq Y \subseteq I \), then \( Y \in F \).

A filter \( F \) is principal if \( F = \{ X \in P(I) : Y \subseteq X \} \) for some \( Y \in P(I) \); otherwise \( F \) is non-principal. A filter \( F \) over \( I \) is \( \kappa \)-complete if for every \( \lambda < \kappa \) and \( \{ X_\alpha \in F : \alpha < \lambda \} \), \( \bigcap_{\alpha < \lambda} X_\alpha \in F \). A filter \( F \) over \( \kappa \) is normal if for every \( \{ X_\alpha \in F : \alpha < \kappa \} \), the diagonal intersection \( \Delta_{\alpha < \kappa} X_\alpha = \{ \xi < \kappa : (\forall \alpha < \kappa)(\xi \in X_\alpha) \} \in F \).

(i) Suppose \( F \) is a normal filter over a cardinal \( \kappa \), \( S \subseteq \kappa, \kappa \setminus S \notin F \), and \( f \) is a regressive function on \( S \). Show that there exists \( X \subseteq S \) and \( \gamma < \kappa \) such that \( \kappa \setminus X \notin F \) and \( (\forall \xi \in X)(f(\xi) = \gamma) \).

(ii) Suppose \( \kappa > cf(\kappa) > \aleph_0 \). The club filter on \( \kappa \) is the family \( C_\kappa = \{ X \in P(I) : X \supseteq C \text{ for some club } C \text{ in } \kappa \} \). Note that \( C_\kappa \) is a filter.

(a) Show that \( C_\kappa \) is \( cf(\kappa) \)-complete.

(b) Prove that if \( \kappa = cf(\kappa) > \aleph_0 \), then \( C_\kappa \) is normal.

5 Splitting Stationary Sets and Solovay’s Theorem

(i) Suppose \( \kappa = cf(\kappa) > \aleph_0 \). Prove that there exists a family \( \{ S_\alpha : \alpha < \kappa \} \) of pairwise disjoint stationary sets such that \( \kappa = \bigcup_{\alpha < \kappa} S_\alpha \). [HINT. Consider cases according as \( \kappa \) is a limit cardinal or a successor cardinal, using some of the elements of an Ulam matrix on \( \kappa \) in the latter case.]

(ii) Optional**: Solovay’s Theorem. Suppose \( S \) is a stationary subset of \( \kappa = cf(\kappa) > \aleph_0 \). Prove that there exists a family \( \{ S_\alpha : \alpha < \kappa \} \) of pairwise disjoint stationary sets such that \( S = \bigcup_{\alpha < \kappa} S_\alpha \).

(iii) Suppose \( S \) is stationary in \( \kappa = cf(\kappa) > \aleph_0 \). Prove that there exists a family \( F \) of \( 2^\kappa \) stationary subsets of \( S \) such that if \( A \neq B \in F \), then \( A \setminus B \) and \( B \setminus A \) are stationary in \( \kappa \). [HINT. Part (ii).]

6 Non-stationary Ideals

An ideal over a non-empty set \( I \) is a family \( N \subseteq P(I) \) such that

(i) \( \emptyset \notin N, I \notin N \);
(ii) if \( X, Y \in N \), then \( X \cup Y \in N \);
Stationary Sets and a Variant of Fodor’s Lemma

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(iii) if \( X \in N, Y \subseteq X \subseteq I \), then \( Y \in F \).

An ideal is an ideal over \( \text{dom}(I) = \bigcup I \). Let \( N^+ = P(I) \setminus N \) be the family of \( N \)-non-negligible sets. The dual filter \( N^* \) of an ideal \( N \) is the filter \( \{X \in P(I) : I \setminus X \in N\} \). An ideal \( N \) is \( \kappa \)-complete and normal if the corresponding dual filter \( N^* \) has these properties. The dual ideal \( F^* \) of a filter \( F \) is defined analogously: \( F^* = \{X \in P(I) : I \setminus X \in F\} \). Clearly, for an ideal \( N \) and a filter \( F \), \( N^{**} = N; F^{**} = F \). For a cardinal \( \kappa \), the non-stationary ideal over \( \kappa \) is the ideal \( NS_\kappa = \{X \in P(I) : X \subseteq Y \) for some non-stationary subset \( Y \subseteq \kappa \}. \) So \( NS_\kappa^+ \) is the collection of stationary sets of \( \kappa \). An ideal \( N \) is \( \lambda \)-saturated if for any \( \{X_\alpha : \alpha < \lambda \} \subseteq N^+ \) there exist \( \beta < \gamma < \lambda \) such that \( X_\beta \cap X_\gamma \in N^+ \). Let \( \text{sat}(N) = \min \{\lambda : N \) is \( \lambda \)-saturated \}.

(i) Show \( C_\kappa^* = NS_\kappa \).

(ii) Prove Ulam’s theorem: there is no \( \kappa^+ \)-saturated \( \kappa^+ \)-complete ideal over \( \kappa^+ \).

(iii) Suppose that for some \( \lambda < \kappa, N \) is a \( \lambda \)-saturated \( \kappa \)-complete ideal over \( \kappa \). Determine whether \( \kappa \) has the tree property, i.e. whether every \( \kappa \)-tree has a cofinal branch. [Hint. WLOG, any candidate \( \kappa \)-tree \( T \) has domain \( T = \kappa \); consider \( D_\xi = \{\zeta \in \kappa : \xi < T \zeta \} \].

7 Stationary Sets and a Variant of Fodor’s Lemma

Let \( A \) be a set of cardinality \( \kappa = \text{cf}(\kappa) > \aleph_0 \). A \( \kappa \)-filtration of \( A \) is an indexed sequence \( \{A_\alpha : \alpha < \kappa \} \) such that for all \( \alpha, \beta < \kappa \)

(a) \( |A_\alpha| < \kappa \);  
(b) \( \alpha < \beta \) implies \( A_\alpha \subseteq A_\beta \);  
(c) \( \delta \in \text{lim}(\kappa) \) implies \( A_\delta = \cup \{A_\alpha : \alpha < \delta \} \);  
(d) \( A = \cup \{A_\alpha : \alpha < \kappa \} \).

(i) Suppose \( \{A_\alpha : \alpha < \kappa \} \) and \( \{B_\alpha : \alpha < \kappa \} \) are \( \kappa \)-filtrations of \( A \). Show the set \( \{\alpha \in \kappa : A_\alpha = B_\alpha \} \) is a club of \( \kappa \).

(ii) Let \( \{A_\alpha : \alpha < \kappa = \text{cf}(\kappa) \} \) be a \( \kappa \)-filtration of \( A \). Prove there exists a club \( C \) of \( \kappa \) such that for all \( \alpha \in C \), \( |A_\alpha \cap A_\alpha^+| = |\alpha^+ \setminus \alpha| \) where \( \alpha^+ \) is the successor of \( \alpha \) in \( C \), i.e. \( \alpha^+ = \inf \{\beta \in C : \beta > \alpha \} \).

(iii) Suppose \( \{A_\alpha : \alpha < \kappa \} \) is a \( \kappa \)-filtration of a set \( A \) of cardinality \( \kappa \). Prove the following variant of Fodor’s Lemma: if \( S \) is a stationary subset of \( \kappa \) and \( f : S \rightarrow A \) is a function such that for all \( \alpha \in S \), \( f(\alpha) \in A_\alpha \), then there exists a stationary \( S' \subseteq S \) such that \( f \upharpoonright S' \) is constant.

8 Clubs and Games

Let \( W \subseteq [\omega_1]^{<\omega_1} \). Let \( G_W \) be the following game of length \( \omega \): players I and II take turns to choose countable ordinals \( \alpha_0, \alpha_1, \ldots \); player I wins \( G_W \) if \( \{\alpha_n : n \in \omega\} \in W \).

Regarding each countable ordinal \( \alpha \) as the set \( \{\beta : \beta < \alpha \} \), show that player I has a winning strategy if and only if \( W \) contains a club of \( \omega_1 \). Show that player II has a winning strategy if and only if the complement of \( W \) contains a club. Deduce that there are games \( G_W \) which are not determined, i.e. neither player has a winning strategy.
9 **Weakly Compact Cardinals**

(i) Let $A$ be a set of cardinals such that for every regular cardinal $\lambda \in A, A \cap \lambda$ is not stationary in $\lambda$. Prove there exists an injective function $g$ on $A$ such that $(\forall \alpha \in A)(g(\alpha) < \alpha)$.

(ii) Suppose that $\kappa$ is a weakly compact cardinal, i.e. $\kappa$ is (strongly) inaccessible and there are no $\kappa$-Aronszajn trees ($\kappa$ has the tree property). Prove that for every stationary subset $S$ of $\kappa$, there is a regular cardinal $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in $\lambda$.

10 **Kueker Clubs**

Suppose $\kappa = \text{cf}(\kappa) > \aleph_0$. A club on $[\kappa]^{<\kappa}$ is a family $C \subseteq [\kappa]^{<\kappa}$ such that $C$ is closed under unions of chains of length less than $\kappa$ and $(\forall X \subseteq [\kappa]^{<\kappa})(\exists Y \in C)(X \subseteq Y)$. A set $S \subseteq [\kappa]^{<\kappa}$ is stationary in $[\kappa]^{<\kappa}$ if $S \cap C \neq \emptyset$ for every club $C$ in $[\kappa]^{<\kappa}$. Formulate and prove analogues of the standard results on clubs, stationary sets and regressive functions for the above definitions.

11 Prove that $\lozenge$ implies that there exists a family $\{Z_\alpha : \alpha < 2^{\aleph_1}\}$ such that:

(i) $(\forall \alpha < 2^{\aleph_1}, Z_\alpha$ is a stationary subset of $\aleph_1$;

(ii) $(\forall \alpha < \beta < 2^{\aleph_1}, Z_\alpha \cap Z_\beta$ is countable.

12 **Inconsistent Bogus Diamonds**

Let $\lozenge(1)$ be the assertion $(\exists\{A_\alpha \subseteq \alpha : \alpha < \omega_1\})(\forall X \subseteq \omega_1)(\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ contains a club in $\omega_1$). Let $\lozenge(2)$ be the assertion $(\exists\{A_\alpha \subseteq \alpha : \alpha < \omega_1\})(\forall X \subseteq \omega_1)(\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary in $\omega_1$).

(i) Show $\lozenge(1)$ is false.

(ii) Show $\lozenge(2)$ is false.

13 **Diamonds for Functions**

Suppose that $S$ is a stationary subset of $\lambda = \text{cf}(\lambda) > \aleph_0$. Let $\lozenge_\lambda(S)$ denote the assertion $(\exists\{A_\alpha \subseteq \alpha : \alpha \in S\})(\forall X \subseteq \lambda)(\{\alpha \in S : X \cap \alpha = A_\alpha\}$ is stationary in $\lambda$). Let $\lozenge_\lambda$ mean $\lozenge_\lambda(\lambda)$; so in this notation $\lozenge$ is $\lozenge_{\omega_1}$.

Prove that $\lozenge_\lambda(S)$ is equivalent to the assertion: there exists $\{f_\alpha : \alpha \in S\}$ such that:

(i) $(\forall \alpha \in S, f_\alpha : \alpha \to \alpha$ is a function;

(ii) $(\forall f : \lambda \to \lambda, \{\alpha \in S : f \upharpoonright \alpha = f_\alpha\}$ is stationary in $\lambda$. 

4
14 DIAMOND EQUIVALENCES

Let $\diamondsuit'$ denote the following assertion: there exists a family \( \{E_\alpha : \alpha \in \omega_1\} \) such that:

(i) \( \forall \alpha \in \omega_1, E_\alpha \) is a countable set of subsets of \( \alpha \);

(ii) \( \forall X \subseteq \omega_1, \{\alpha \in \omega_1 : X \cap \alpha \in E_\alpha\} \) is stationary in \( \omega_1 \).

Prove that $\diamondsuit'$ and $\diamondsuit$ are equivalent in ZFC.

15 DIAMOND AND THE CLUB PREDICTION PRINCIPLE

(i) Show that $\diamondsuit$ implies $\clubsuit$.

(ii) Prove Devlin’s theorem (1979): $\clubsuit + CH$ implies $\diamondsuit$. Deduce that $\diamondsuit$ is equivalent to $\clubsuit + CH$.

16 STRONGER DIAMONDS

Let $\diamondsuit^+$ denote the assertion there exists \( \{S_\alpha \subseteq P(\alpha) : \alpha \in \omega_1\} \) such that \( |S_\alpha| \leq \aleph_0 \) and \( (\forall X \subseteq \omega_1)(\exists B \in [\omega_1]^{\omega_1})(\forall \alpha < \omega_1)(\alpha = \sup(B \cap \alpha) \Rightarrow X \cap \alpha \in S_\alpha \land B \cap \alpha \in S_\alpha) \).

(i) Prove that $\diamondsuit^+$ implies $\diamondsuit'$.

(ii) Deduce that $\diamondsuit^+$ implies $\diamondsuit$.

REMARK: Jensen proved that $\diamondsuit^+$ implies the Kurepa Hypothesis. This stronger diamond can be defined for the other uncountable cardinals \( \kappa \) and used to settle combinatorial hypotheses about \( \kappa \)-trees.

17 OPTIONAL. INEFFABLE CARDINALS AND THE \( \kappa \)-KUREPA HYPOTHESIS

A cardinal \( \kappa \) is ineffable if \( \kappa = cf(\kappa) > \aleph_0 \) and whenever \( f : [\kappa]^2 \rightarrow 2 \) is a function, then there is a stationary set \( S \subseteq \kappa \) such that \( f \upharpoonright S \) is constant, i.e. \( (\exists \gamma)(\forall \xi \in S)(f(\xi) = \gamma) \).

(i) Let \( \kappa = cf(\kappa) > \aleph_0 \). Suppose (*) holds: for every \( \{A_\alpha \subseteq \alpha : \alpha < \kappa\} \), there exists \( A \subseteq \kappa \) such that \( \{\alpha < \kappa : A \cap \alpha = A_\alpha\} \) is stationary in \( \kappa \). Prove that \( \kappa \) is ineffable.

(ii) Prove that if \( \kappa \) is ineffable, then (*) holds.

(iii) Prove the following result of Jensen and Kunen: if \( \kappa \) is ineffable, then the \( \kappa \)-Kurepa Hypothesis is false. [HINT. Do this by refuting the existence of a \( \kappa \)-Kurepa family, i.e. a family \( F \subseteq P(\kappa) \) such that \( |F| = \kappa^+ \) and \( (\forall \alpha < \kappa)(\{X \cap \alpha : X \in F\} \) has cardinality at most \( |\alpha| + \aleph_0\); it is a well-known short theorem that the existence of such a family is equivalent to that of a \( \kappa \)-Kurepa tree.]

18 (i) Suppose \( \pi \) is a bijection from \( \lambda^+ \geq \aleph_1 \) onto \( \lambda \times \lambda^+ \). Show there exists a club \( C \) of \( \lambda^+ \) such that for all \( \delta \in C \), the restriction map \( \pi \upharpoonright \delta \) is a bijection from \( \delta \) onto \( \lambda \times \delta \).

(ii) For a cardinal \( \lambda \) and a set \( W \), let \( [W]^{\leq \lambda} = \{Y \subseteq W : |Y| \leq \lambda\} \). If \( 2^\lambda = \lambda^+ \), let \( \{X_\alpha : \alpha < \lambda^+\} \) be an enumeration of \( [\lambda^+]^{\leq \lambda} \) and suppose \( Z \subseteq \lambda^+ \). Show that for some club \( C \) of \( \lambda^+ \), for all \( \delta \in C \) there are arbitrarily large \( \alpha < \delta \) such that for some \( \beta < \delta, Z \cap \alpha = X_\beta \).

(iii) Suppose \( cf(\delta) = \kappa > \aleph_0 \) and \( h \) is a function from \( \text{dom}(h) \supseteq \delta \) into \( \kappa \). Prove that the following are equivalent:
(a) \( h \) is one-to-one on some club \( C \) of \( \delta \);
(b) \( h \) is strictly increasing on some club \( D \) of \( \delta \);
(c) \( \text{range}(h \upharpoonright S) \) is unbounded in \( \kappa \) for every stationary subset \( S \subseteq \delta \).

**Remark** These propositions are the first steps of a very recent short proof by Peter Komjath of Shelah’s theorem (see the reference below) that \( 2^\lambda = \lambda^+ \) implies \( \diamondsuit_\lambda \) for \( \lambda \geq \aleph_1 \); this is not the case for \( \lambda = \aleph_0 \) as \( \text{CH} \) does not imply \( \diamondsuit \).

**Open Research Problems.**

(i) Assume that \( \lambda = \lambda^{<\lambda} = 2^\mu \) is a regular limit cardinal. Determine whether \( \diamondsuit_\lambda \) is a theorem of \( \text{ZFC} \).