This Example Sheet involves questions that require significant extensions of the material covered in lectures. Most are optional; on the other hand, several are core results, which are chosen to illustrate the richness and flexibility of forcing and set-theoretic methods.

1 Concerning forcings, anti-chains and generic sets

(i) Prove that a set \( G \subseteq P \) is generic over \( M \) if and only if for every maximal anti-chain \( A \in M \) of \( P \setminus G \cap A \setminus = 1 \). [HINT. One direction uses AC.]

(ii) A set \( D \subseteq P \) is:

(a) pre-dense above \( p \in P \) if \( (\forall q \in P)(q \geq p \rightarrow (\exists d \in D)(d \text{ and } q \text{ are compatible})) \); \( D \) is pre-dense if \( D \) is pre-dense above \( 0_M \);

(b) dense above \( p \in P \) if \( (\forall q \in P)(q \geq p \rightarrow (\exists d \in D)(d \geq q)) \). So \( D \) is dense in \( P \) if \( D \) is dense above \( 0_M \).

Suppose that \( E \) is pre-dense in \( P \) and \( G \subseteq P \) is generic over \( M \). Show that \( G \cap E \neq \emptyset \).

Suppose that \( E \) is pre-dense above \( q \in P \) and \( G \subseteq P \) is generic over \( M \). Show that if \( q \in G \), then \( G \cap E \neq \emptyset \).

(iii) Deduce that the following are equivalent for a directed downward closed set \( G \subseteq P, P \in M \) where \( M \) is a transitive model of \( ZFC \).

(a) \( G \) is generic in \( P \) over \( M \);

(b) \( G \cap D \neq \emptyset \) for every dense open set \( D \subseteq P \) in \( M \);

(c) \( G \cap C \neq \emptyset \) for every dense subset \( C \subseteq P \) in \( M \);

(d) \( G \cap B \neq \emptyset \) for every pre-dense subset \( B \subseteq P \) in \( M \);

(e) \( G \cap A \neq \emptyset \) for every maximal anti-chain \( A \subseteq P \) in \( M \).

(iv) Suppose that \( M \) is a CTM of \( ZFC, P \in M, E \subseteq P, E \in M, \) and \( G \) is generic over \( M \). Prove that either \( G \cap E \neq \emptyset \) or \( (\exists q \in G)(\forall r \in E)(r \text{ and } q \text{ are incompatible}) \). [HINT. Consider \( \{ p \in P : (\exists r \in E)(r \leq p) \} \cup \{ q \in P : (\forall r \in E)(r \text{ and } q \text{ are incompatible)} \} \in M. \)]

(v) Suppose \( M \) is a CTM and \( P \in M \) is a separative forcing. Prove that there are \( 2^{\aleph_0} \) generic sets in \( P \) over \( M \).

2 The Forcing Relation

Suppose that \( P \) is a non-trivial forcing, \( p, q \in P \), and \( \varphi \) is a formula in the vocabulary of \( ZFC \) which may contain \( P \)-names. Show:
Suppose $G \subseteq P$ is generic over $\mathbb{P}$.

(i) Suppose $\sigma, \tau \in M^P$. Show $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

(ii) Suppose $\tau \in M^P$ and $\text{range}(\tau) \subseteq \{\dot{n} : n \in \omega\}$. Let $\sigma = \{\langle p, \dot{n} \rangle : (\forall q \in P)(\langle q, \dot{n} \rangle \in \tau \rightarrow p \perp q)\}$. Show that $\sigma_G = \omega \setminus \tau_G$. [HINT. The set $\{r \in P : (\exists p \leq r)(\langle p, \dot{n} \rangle \in \sigma \lor (p, \dot{n}) \in \tau)\}$ is dense.]

(iii) Suppose $A$ is an anti-chain in $\mathbb{P}$ and for each $a \in A$, $\tau_a$ is a $\mathbb{P}$–name. Show there exists a $\mathbb{P}$–name $\tau$ such that for every $a \in A$, if $a \in G$, then $\tau[G] = \tau_a[G]$, and $\tau[G] = \emptyset$ if $G \cap A = \emptyset$. [HINT. Suppose $\tau_a = \{\langle q_{a,j}, \tau_{a,j} \rangle : j < i\}$. Consider the $\mathbb{P}$–name $\tau = \{(r, \tau_{a,j}) : a \in A, j < i_a, r \geq q_{a,j}, \text{ and } r \geq a\}$; refer to Question 1.]

4 Nice Names and Bounds for the Continuum

Suppose $\mathbb{P} \in \mathbb{M}$ and $G \subseteq P$ is generic over $\mathbb{M}$. A name $\tau \in M^P$ is a nice name for a subset $x$ of $\sigma \in M^P$ if $\tau = \bigcup \{A_{\pi} \times \{\pi\} : \pi \in \text{range}(\sigma)\}$, where $A_{\pi}$ is an anti–chain in $\mathbb{P}$.

(i) Prove that for all $\sigma, \rho \in M^P$ there exists a nice name $\tau$ such that $\|\tau\| (\rho \subseteq \sigma \rightarrow \rho = \tau)$. [HINT. For $\pi \in \text{range}(\sigma)$, let $A_{\pi}$ be maximal relative to the properties (1) $(\forall p \in A_{\pi})(p \perp \pi \in \rho)$ and (2) $A_{\pi}$ is an anti–chain in $\mathbb{P}$; refer to a previous Question to check that $\tau$ as defined works.]

(ii) Suppose ($\mathbb{P}$ is a c.c.c. forcing and $\lambda$ is a cardinal)$^M$. Let $\kappa^* = (\langle \mathbb{P} | \lambda \rangle)^M$. Then $(2^{\lambda} \leq \kappa^*)^{M[G]}$. [HINT. Count the number of nice names for the members of $P(\lambda)^{M[G]}$, remembering that $\mathbb{P}$ has the countable chain condition.]

(iii) Deduce that if ($\lambda$ is a cardinal and $\lambda^{\aleph_0} = \lambda)^M$, then there is a generic extension $\mathbb{M}[H]$ such that $(2^{\aleph_0} = \lambda)^{M[H]}$. 

\[2\]
5 Adding Cohen Reals and Suslin Trees

(i) A tree $T$ is ever-branching if for every $s \in T$, the set $\{ t \in T : s \leq_T t \}$ is not linearly ordered. Let $M$ be a CTM such that $(T$ is an ever-branching Suslin tree$)^M$. Suppose that $(\mathbb{P} = Fn(\lambda \times \omega, 2, \aleph_0) \land \lambda \geq \aleph_0)^M$. Prove that for any set $G \subseteq P$ generic over $M$, $M[G] \models (T$ is a Suslin tree$)$.

(ii) Deduce that there is a model of $ZFC$ in which there is a Suslin tree but $CH$ fails.

Remark: So the existence of Suslin trees does not imply $CH$ (nor a fortiori $\diamondsuit$). It is a theorem of Shelah that adding a Cohen real creates a Suslin tree in the generic extension. The proof that $CH$ does not imply Suslin trees is due to Jensen.

6 (i) Prove that the theory $ZFC + \diamondsuit$ is relatively consistent. [HINT. It may be easier to verify $\diamondsuit$ in its functional form (see Example Sheet 3). Let $I = \{ \langle \alpha, \zeta \rangle : \zeta < \alpha < \omega_1^M \}$ and consider the forcing $Q = Fn(I, 2, \aleph_1^M)$. Show that $Q$ is countably complete, and that if $G$ is generic over $M$, then in $M[G]$, a $\diamondsuit$-sequence is provided by $\langle (\bigcup G)_\alpha : \alpha < \omega_1 \rangle$. For this, noticing that $Q$ adds no new $\omega$-sequences and $\aleph_1^M = \aleph_1^M[G]$, define a sequence of ordinals and conditions forcing an arbitrary club to intersect the family of guesses for a function $f : \omega_1 \to \omega_1$. (Refer to Kunen, Set Theory, chapter VII, or Shelah, Proper and Improper Forcing, chapter 1, if difficulties arise. Remark: $\diamondsuit$ is true in $L$; this is the earliest proof of its consistency, due to Jensen.)

(ii) Deduce that $\diamondsuit$ is independent of $ZFC$.

(iii) Show if $(\lambda$ is a cardinal and $\lambda^\lambda = \lambda$ and $\diamondsuit)^M$, then there is a generic extension $M[H]$ such that $M[H] \models (2^\aleph_0 = \lambda$ and there is a Suslin tree).

(iv) Suppose that $M \models (\mathbb{P}$ is a c.c.c. forcing and $\models \mathbb{P} \leq \aleph_1$ and $\diamondsuit$). Prove that for every $G \subseteq P$ generic over $M$, $M[G] \models \diamondsuit$. [HINT. In $M$, use $\diamondsuit$ to guess nice names for subsets of $\omega_1$].

(v) Deduce that $\diamondsuit$ does not imply $V = L$.

(vi) Suppose that $M \models (\mathbb{P}$ is a c.c.c. forcing) and $M[G] \models \diamondsuit$, where $G \subseteq P$ is generic over $M$. Show that $M \models \diamondsuit$. [HINT. Recall the equivalent characterisations of $\diamondsuit$ from Example Sheet 3 and the lemma about approximating functions in c.c.c. generic extensions.]

(vii) Optional Prove that $\clubsuit$ is independent of $ZFC$.

7 The Levy Collapse

The Levy collapse is the forcing $L\nu(\kappa)$ defined as follows:

$$\{ p : p$ is a finite function, $\text{dom}(p) \subseteq \kappa \times \omega$, and $(\forall \langle \alpha, n \rangle \in \text{dom}(p))(p(\alpha, n) \in \alpha) \}$$
with the partial ordering \( p \leq q \) iff \( p \subseteq q \).

Suppose that \( \kappa \) is strongly inaccessible in the countable transitive model \( \mathbb{M} \).

(i) What is the cardinality of a maximal anti-chain in \( L^\kappa(\kappa) \)?

(ii) What is the largest cardinal \( \lambda \) for which \( L^\kappa(\kappa) \) is \( \lambda \)-closed?

(iii) Which cardinals are preserved and which are collapsed under forcing with \( L^\kappa(\kappa) \)?

(iv) Let \( H \) be generic in \( L^\kappa(\kappa) \) over \( \mathbb{M} \). Prove that \( \kappa = \aleph_{\mathbb{M}[H]}^1 \).

8 Trees

(i) Suppose \( T \) is a tree. For \( \alpha < \text{ht}(T) \) let \( \prec_\alpha \) be a linear ordering of \( T_\alpha = \{ x \in T : \text{ht}(x) = \alpha \} \). Define a binary relation \( \prec_{\text{lex}} \) on \( T \) as follows: for \( x \neq y \in T \), let \( x \prec_{\text{lex}} y \) iff either \( x <_T y \) or \( p_\alpha(x) <_\alpha p_\alpha(y) \), where \( p_\zeta(z) = T_\zeta(z) \cap T_\zeta \) for \( \zeta \leq \text{ht}(z) \) and \( \alpha \) is the least ordinal such that \( p_\alpha(x) \neq p_\alpha(y) \). Show that \((T, \prec_{\text{lex}})\) is a linearly ordered set.

(ii) For \( \delta \in \text{Ord} \) and a linear order \( I \), let \( ^{<\delta} I \) be the family of \( I \)-valued sequences of length less than \( \delta \). Partially order \( ^{<\delta} I \) by \( a \preceq b \) iff \( b \) extends \( a \). Show \( ^{<\delta} I = (^{<\delta} I, \preceq) \) is a tree. What are \( \text{ht}^{<\delta} I \) and \( \text{Lev}_\alpha(^{<\delta} I) \) for \( \alpha < \delta \)?

The tree \( ^{<\delta} I \) is called the complete (or full) \( I \)-ary tree of height \( \delta \).

(iii) Suppose \( \kappa = \aleph_0 \) or \( \kappa \) is strongly inaccessible. Prove that the complete binary tree of height \( \kappa \) is a \( \kappa \)-tree with at least \( \kappa^+ \) paths (i.e. chains which intersect \( \text{Lev}_\alpha(^{<\kappa} 2) \) for each \( \alpha < \kappa \)).

9 Kurepa Trees

(i) Suppose \( \kappa \) is an infinite cardinal. A tree \( T \) is a \( \kappa \)-Kurepa tree iff \( T \) is a \( \kappa \)-tree with at least \( \kappa^+ \) paths. The \( \kappa \)-Kurepa Hypothesis (\( \kappa \)-KH) is the assertion that there exists a \( \kappa \)-Kurepa tree. Kurepa’s Hypothesis (\( KH \)) is \( \aleph_1 \)-KH, i.e. there exists an \( \aleph_1 \)-tree with at least \( \aleph_2 \) paths.

Deduce that \( \kappa \)-KH holds trivially if \( \kappa = \aleph_0 \) or \( \kappa \) is strongly inaccessible.

(ii) A \( \kappa \)-Kurepa family is a family \( F \subseteq P(\kappa) \) such that

\[
| F | \geq \kappa^+ \text{ and } (\forall \alpha < \kappa)(| \{ A \cap \alpha : A \in F \} | < \kappa).
\]

Prove that for any regular cardinal \( \kappa \), there is a \( \kappa \)-Kurepa family iff there exists a \( \kappa \)-Kurepa tree.

(iii) Suppose \( \kappa \) is strongly inaccessible in the CTM \( M \) and let \( H \) be generic in \( L^\kappa(\kappa) \) over \( \mathbb{M} \). Prove that \( KH \) holds in the generic extension \( M[H] \).

[HINT. Part (i) above and the part (iii) of the previous question.] Comments: \( \text{Con}(\text{ZFC} + KH) \) can be proved from \( \text{Con}(\text{ZFC}) \) but a more complicated forcing is involved (see Kunen, chapter VII, Exercise H19). However,
Con(ZFC + \neg KH), which requires an iterated forcing argument, does require the hypothesis of the existence of a strongly inaccessible cardinal, because \neg KH implies \aleph_2 is inaccessible in L (see Kunen, Chapter VII, Exercise B9).

10 ARONSZAJN TREES

(i) Suppose \kappa \geq \aleph_0. Show that every \kappa–Suslin tree is a \kappa–Aronszajn tree (i.e. a \kappa–tree whose every chain is of cardinality less than \kappa).

(ii) Prove there exists an Aronszajn tree, i.e. an \aleph_1–Aronszajn tree. [HINT. Consider a subtree of \{ f \in \subset^\omega \omega : f \ is \ injective \}. We know that there are no \aleph_0–Aronszajn trees by König’s Lemma.]

(iii) Show that there exist functions \{ e_\alpha : \alpha < \omega_1 \} such that each \ e_\alpha : \alpha \to \omega \ is injective and for \beta < \alpha, e_\beta \ | \beta \ are identical at all but finitely many points. [HINT. Define the functions by induction on \alpha < \omega_1; at successor ordinals, chose a one-point extension; at a limit ordinal \delta, write \delta = \sup_{n<\omega} \alpha_n where \alpha_0 < \alpha_1 < \ldots \ and \ use \ the \ functions \ e_\alpha_n.]

(iv) Now prove there exists an Aronszajn tree. [HINT. Consider the tree \ T = \{ e_\alpha \ | \beta : \beta < \alpha < \omega_1 \} with the functions from above partially ordered by \ f < g \ iff \ g \ is \ a \ proper \ extension \ of \ f.]

11 OPTIONAL: PROPERTIES UNDER PRODUCTS OF TREES

Suppose that for \ i = 1, 2, \ T_i \ is \ a \ \kappa–tree \ and \ \lambda_i, \kappa \ are \ infinite \ cardinals. Let the product tree \ T_1 \times T_2 = (\cup_{\alpha<\lambda}(Lev_\alpha(T_1) \times Lev_\alpha(T_2)), \subset_{T_1 \times T_2}) be defined as follows: \ (x, y) \subset_{T_1 \times T_2} (u, v) \ iff \ x \subset_{T_1} u \ and \ y \subset_{T_2} v.

(i) Verify that \ T_1 \times T_2 \ is \ a \ \kappa–tree.

(ii) Find the optimal value \mu \ (if \ it \ exists) \ as \ a \ function \ of \ \lambda_i \ such \ that \ if \ T_i \ is \ or \ has \ the \ property \ \lambda_i–P \ (and \ this \ makes \ sense), \ i = 1, 2, \ then \ the \ product \ T_1 \times T_2 \ is \ or \ has \ the \ property \ \mu–P, \ where \ the \ property \ is \ listed \ below: \ (a) \ the \ \mu–chain \ condition; \ (b) \ \mu–Suslin; \ (c) \ \mu–Aronszajn; \ (d) \mu–Kurepa.

12 OPTIONAL: PROPERTIES UNDER FORMATION OF SUBTREES

Suppose \lambda, \kappa \ are \ infinite \ cardinals. Of the following properties, determine which are preserved under formation of subtrees: \ (a) \ \kappa–arboreality, \ i.e. being a \ \kappa–tree; \ (b)well-prunedness; \ (c) normality; \ (d) the \ \lambda–chain condition; \ (e) \ \lambda–Suslin; \ (f) \ \lambda–Aronszajn; \ (g) \ \lambda–Kurepa.

13 OPTIONAL: PROPERTIES UNDER FORMATION OF DIRECTED UNIONS

Suppose \lambda, \kappa \ are \ infinite \ cardinals. Which, if any, of the following properties are preserved under unions of directed chains of \kappa–trees of length \delta \geq \omega ? I.e. if for \ i < j < \delta, \ T_i \ is \ a \ subtree \ of \ T_j \ and \ every \ T_i \ has \ the \ stated \ property, \ then \ \cup_{i<\delta} T_i \ also \ has \ the \ stated \ property \ (whenever \ this \ makes \ sense). \ (a) \ well-prunedness; \ (b) normality; \ (c) the \ \lambda–chain condition; \ (d) \ \lambda–Suslin; \ (e) \ \lambda–Aronszajn; \ (f) \ \lambda–Kurepa.

5
14 Tall Trees that are Not Too Fat Have Long Branches

(i) Suppose that $\kappa$ and $\lambda$ are regular cardinals, $\kappa < \lambda$, and $T = (T, \leq_T)$ is a $\lambda$-tree, each of whose levels has cardinality less than $\kappa$. Using Fodor’s Lemma or otherwise, prove that $T$ has a cofinal branch. (It may help to assume that $T = \lambda$, i.e. the universe of the tree is the set of ordinals less than $\lambda$.)

(ii) Determine whether the above result holds if $\kappa$ is singular.

15 The Axiom of Choice in $L$

(i) Define by transfinite recursion a $\Delta^1_1$ well-ordering $<_\alpha$ of $L_\alpha$ for $\alpha \in \text{Ord}$ and a $\Sigma^1_1$ well-ordering $<_L$ of $L$. [HINT. The case $<_\delta$ for limit $\delta$ is immediate (take unions). For the successor case $<_\beta + 1$, recall that the codes of formulas without parameters can be identified with elements of $L_\omega$ and can be well-ordered using the lexicographic order; since the parameters of a set in $L_\beta + 1$ arise already in $L_\beta$, they can be well-ordered by $<_\beta$. So the predicate $x <_L y \iff (\exists \alpha)(x \in L_\alpha \land x <_\alpha y)$ has the required complexity.]

(ii) Deduce that $V = L$ implies $AC$.

(iii) Conclude that $L$ is a model of $ZFC$.

16 Zermelo, Strong Inaccessibility and Second-Order Set Theory

(i) Suppose $\kappa = cf(\kappa) > \aleph_0$. Prove that $\kappa$ is a strongly inaccessible cardinal if and only if $V_\kappa = H_\kappa$.

(ii) Suppose $\kappa = cf(\kappa) > \aleph_0$. Show that if the class $H_\kappa$ of sets of cardinality hereditarily less than $\kappa$ is a model of $ZFC$, then $\kappa$ is a strongly inaccessible cardinal.

(iii) Let $ZFC^2$ denote the second-order axiomatic system obtained from $ZFC$ in which the Replacement Schema is expressed as the following single axiom with a universal second-order quantifier: $(\forall C)((\forall z \forall u \forall v)((z,v) \in C \land (z,u) \in C \Rightarrow v = u) \Rightarrow \exists x \forall y (y \in x \iff \exists z (z \in a \land (z,y) \in C))$, i.e. if $C$ is a functional class and $a$ is a set, then $\{C(z) : z \in a\}$, the image of $a$ under $C$, is also a set. The intended interpretation of the second-order variables ranges over arbitrary subsets of the domain. Notice that models of $ZFC^2$ are well-founded. Show that if $\kappa$ is a cardinal such that $V_\kappa \models ZFC^2$, then $\kappa$ is strongly inaccessible.

(iv) For each strongly inaccessible cardinal $\kappa$, there is up to isomorphism exactly one model of $ZFC^2$ with the set of ordinals of order type $\kappa$, namely, $(V_\kappa, \in)$. [HINT. Mostowski Collapse; induction on rank.]
17 The Generalized \( \Delta \)-System Lemma

(i) Suppose that \( \lambda < \kappa = cf(\kappa) \) and \( \bigwedge_{\alpha < \kappa} \alpha^\lambda < \kappa \). Prove that if \( |A| = \kappa \) and \( x \in A \) implies \( |x| < \lambda \), then there exists \( B \subseteq A \) such that \( |B| = \kappa \) and \( (\exists \rho)(\forall x \in B)(x \neq y \rightarrow x \cap y = r) \). \[ \text{HINT. This is a standard result. WLOG, } \bigcup A \subseteq \kappa \text{ and some fixed } \rho < \lambda \text{ is the order type of every } x = \langle x(\xi) : \xi < \rho \rangle \in A. \text{ Using } \bigwedge_{\alpha < \kappa} \alpha^\lambda < \kappa \text{ and } \kappa = cf(\kappa), \text{ let } \xi_0 \text{ be the minimal } \xi \text{ such that } \{x(\xi) : \xi \in A\} \text{ is cofinal in } \kappa; \text{ let } \sigma = sup\{x(\eta) + 1 : x \in A \land \eta < \xi_0\}, \text{ so } (\ast) x \upharpoonright \xi_0 \subseteq \sigma < \kappa; \text{ now define by induction } \{x(\alpha) : \alpha < \kappa\} \text{ such that } x_\alpha(\xi_0) > max\{\sigma, sup\{x_\beta(\eta) : \beta < \alpha \land \eta < \rho\}\}. \text{ Use } (\ast) \text{ and } \sigma^\lambda < \kappa \text{ to refine } \{x(\alpha) : \alpha < \kappa\} \text{ and extract a root } r \subseteq \sigma. \]

(ii) Find a family of \( \mathcal{R}_\omega \) finite sets such that no subfamily of size \( \mathcal{R}_\omega \) has a root.

18 Optional: Moderately Large Cardinals Do Not Decide CH

A cardinal \( \kappa \) is called Ramsey if \( \kappa \rightarrow (\kappa)^2_\omega \). In 1961, Scott proved that the existence of large enough cardinals contradicts the axiom \( V = L \). However, the Continuum Hypothesis appears to behave differently.

(i) Show that the set \( \lambda^2 \) with the lexicographic order \( \preceq_{\text{lex}} \) contains no increasing or decreasing sequences of length \( \lambda^+ \). \[ \text{HINT. Otherwise suppose } H \text{ is e.g. } \preceq_{\text{lex}} \text{-increasing of size } \lambda^+; \text{ WLOG, } H = \{h_\alpha : \alpha < \lambda^+\} \text{ and for some least } \gamma \leq \lambda, \forall g, h \in H, |g \upharpoonright \gamma \neq h \upharpoonright \gamma|. \text{ Now find } \xi^* < \gamma \text{ such that } \{h_\alpha : \xi^* : \alpha < \lambda^+\} \text{ has cardinality } \lambda^+. \]

(ii) Prove that \( 2^\lambda \rightarrow (\lambda^+)^2_\omega \). \[ \text{HINT. Otherwise, consider a homogeneous set } Y \text{ of size } \lambda^+ \text{ for the } 2\text{-colouring } F \text{ of } [\lambda^2]_2 \text{ given by } F\{f, g\} = 0 \text{ if and only if } f \preceq_{\text{lex}} g. \text{ This is due to Sierpiński and Kurepa independently.} \]

(iii) Prove that if \( \kappa \) is a Ramsey cardinal, then \( \kappa \) is strongly inaccessible. \[ \text{HINT. Regularity is easy: use the colouring } c(\{\alpha, \beta\}) = 0 \leftrightarrow (\exists \xi)(\{\alpha, \beta\} \subseteq X_\xi) \text{ where } \kappa = \bigcup_{\xi < \gamma} X_\xi, |X_\xi| < \kappa; \text{ for strong inaccessibility, use the previous part.} \]

(iv) Prove that \( \kappa \) is a Ramsey cardinal if and only if for all \( \beta < \kappa, \kappa \rightarrow (\kappa)^{<\omega}_\beta \). \[ \text{HINT. For the hard direction, if } f : [\kappa]^{<\omega} \rightarrow \beta, \text{ define a 2-colouring } g \text{ as follows: } g(\{\xi_1, \ldots, \xi_n\}) = 0 \leftrightarrow n = 2m \land f(\{\xi_1, \ldots, \xi_m\}) = f(\{\xi_{m+1}, \ldots, \xi_{2m}\}). \text{ Notice if } H \text{ is homogeneous for } g \text{ and of cardinality } \kappa, \text{ then } g \upharpoonright |H|^n \equiv 0 \text{ and so } H \text{ is homogeneous for } f \text{ also.} \]

(v) Suppose that \( M \models (\kappa \text{ is a Ramsey cardinal and } \mathbb{P} \text{ is a forcing of cardinality less than } \kappa) \). Let \( G \) be \( \mathbb{P} \)-generic over \( M \). Prove that \( M[G] \models (\kappa \text{ is a Ramsey cardinal}). \[ \text{HINT. If } M[G] \models (\tau_G \text{ is a colouring of } [\kappa]^{<\omega}, \text{ and this statement is forced by some condition } p \in G, \text{ consider the colouring } g : [\kappa]^{<\omega} \rightarrow P(\mathbb{P} \times 2) \text{ defined by } g(a) = \{\langle q, x \rangle : p \leq q \wedge q \not\models (\tau(a) = \dot{x})\}. \text{ Note that } g \in M \text{ and } P(\mathbb{P} \times 2) \text{ has cardinality } \beta \text{ for some } \beta < \kappa \text{ (by strong inaccessibility). Use the previous part, in } M, \text{ to obtain a homogeneous set } Y \in M; \text{ show } p \not\models (Y \text{ is homogeneous for } \tau). \]
(vi) Deduce that \( CH \) is independent of \( ZFC + (\exists \kappa)(\kappa \text{ is a Ramsey cardinal}) \).

REMARK. This type of result in very general form is due to Levy and Solovay (1967); see A. Kanamori, The Higher Infinite, Springer, 2009. The power of large cardinals to decide a statement \( \varphi \) is thus circumscribed by the existence of forcings of relatively small size, if such forcings can be used to establish the relative independence of \( \varphi \) from \( ZFC \).

19 Optional: Embeddings and Elementary Embeddings of \( V \)

An uncountable cardinal \( \kappa \) is measurable if there exists a \( \kappa \)-complete non-principal ultrafilter \( U \) over \( \kappa \). Suppose that \( M \) is a (transitive) model of \( ZFC \). While the product structure \( M^\kappa \) does not satisfy much of \( ZFC \), the ultrapower \( M^\kappa / U \) of \( M \) relative to \( U \) is a model of \( ZFC \). Its elements are the equivalence classes \( (f)_U \) of the relation \( \equiv_U \) defined by \( g \equiv_U h \) if \( \{ \alpha < \kappa : g(\alpha) = h(\alpha) \} \in U \) for \( g, h \in M^\kappa \). (Care is needed in the case where \( M \) is a proper class; one uses the restricted equivalence classes consisting of elements of least rank, a stratagem called Scott’s trick.) The mapping \( \iota_U : M \to M^\kappa / U \) defined by \( \iota_U(a) = (a^*_U) \) where \( a^*(\alpha) = a \) for all \( \alpha < \kappa \) is an elementary embedding, and so \( M^\kappa / U \) is a model of \( ZFC \). If \( U \) is \( \aleph_1 \)-complete, then \( M^\kappa / U \) is also well-founded and satisfies the hypotheses of Mostowski’s Lemma. Its transitive collapse under the Mostowski collapsing map \( \pi \) is denoted \( M_U \). Taking \( M = V \), the associated mapping \( j_U : V \to V_U \), defined by \( j_U(a) = \pi(\iota_U(a^*_U)) \) is an elementary embedding of \( V \) into \( V_U \).

(i) Scott’s Theorem; 1961

Show that if there is a measurable cardinal, then \( V \neq L \). [HINT. Suppose that \( \kappa \) is a measurable cardinal; what is the first ordinal moved by \( j_U \)? If \( \kappa \) is the smallest measurable cardinal, what will it be in the ultrapower?]

(ii) Kunen’s Theorem; 1971

Prove that no non-trivial elementary embedding exists from \( V \) into \( V \). [HINT. Otherwise, use question 7(iv) from Example Sheet 3 to derive a contradiction; for details, see A. Kanamori, The Higher Infinite, Springer, 2009, pages 319–320.]

REMARK. Joel Hampkins has shown very recently that every countable model of set theory \( (M, \in^M) \), including every well-founded model, is isomorphic to a submodel of its own constructible universe \( (L^M, \in^L) \). In terms of embeddings (i.e. injective homomorphisms), there is an embedding \( j : (M, \in^M) \to (L^M, \in^L) \) that is elementary for quantifier-free assertions in the language of set theory. See: J. D. Hampkins, J. Math. Log., 13, 1350006 (2013) [27 pages].

20 Optional: Forcing and Partial Isomorphisms

Suppose that \( A \) and \( B \) are \( \tau \)-structures in a vocabulary \( \tau \). Say that \( A \) and \( B \) are partially isomorphic, denoted \( A \simeq_p B \) if some non-empty family \( F \subseteq \text{PART}(A, B) \) of the partial isomorphisms from \( A \) to \( B \) is a back-and-forth set for \( A \) and \( B \):

\[
(\forall f \in F)(\forall a \in A)(\exists g \in F)(f \subseteq g \land a \in \text{dom}(g)) \quad \text{and} \quad (\forall f \in F)(\forall b \in B)(\exists g \in F)(f \subseteq g \land b \in \text{range}(g)).
\]
(i) Prove that if $\tau, A, B$ are countable and $A \simeq_p B$, then $A \simeq B$.

(ii) Show that the converse of (i) fails. [HINT. Consider the linear orders $\mathbb{Q}$ and $\mathbb{R}$.]

(iii) Prove that if two structures $A$ and $B$ are partially isomorphic, then there is a forcing extension in which they are isomorphic.

REMARK. Partial isomorphism yields a characterization of elementary equivalence in the infinitary language $L_{\infty, \omega}$. For a recent introduction to these ideas, see J. Väänänen, *Models and Games*, Cambridge University Press, 2011.

21 Optional: Hereditarily Countable Sets and the Power Set Axiom

Let $ZFC^-S$ be the theory $ZFC$ without the Power Set axiom.

(i) Suppose $\kappa = cf(\kappa) \geq \aleph_1$. Prove that $H_\kappa$ is a model of $ZFC^-S$.

(ii) Deduce that no proof of the existence of $\mathbb{R}$ avoids some non-trivial use of the Power Set axiom.

22 Optional: Forcing, Chain Conditions, and Elementary Submodels

For a forcing $\mathbb{P}$, a cardinal $\kappa$ is *large enough* (for $\mathbb{P}$) if $\kappa = cf(\kappa) > \aleph_1$ and the set of dense subsets of $\mathbb{P}$ belongs to $H_\kappa$ (so in particular, $\mathbb{P}$, the conditions in $\mathbb{P}$ and every dense subset of $\mathbb{P}$ all belong to $H_\kappa$). For a set $N$, a condition $p \in \mathbb{P}$ is called *$N$–generic* if for every $D \in N$ which is a dense subset of $\mathbb{P}$, $D \cap N$ is pre-dense above $p$.

Suppose that $\kappa$ is large enough for $\mathbb{P}$. Prove the following are equivalent:

(i) $\mathbb{P}$ has the countable chain condition;

(ii) for every countable elementary submodel $N$ of $H_\kappa$, $0_\mathbb{P}$ is $N$–generic;

(iii) every countable subset $X$ of $H_\kappa$ is contained in an elementary submodel $N$ of $H_\kappa$ such that $0_\mathbb{P}$ is $N$–generic.

[HINT. For $(1) \Rightarrow (2)$, consider an $A \in N$ maximal relative to the property of being an anti-chain contained in $D$. For $(3) \Rightarrow (1)$, show if $A \in N$ is a maximal anti-chain, then $\overline{A} = \{p \in P : (\exists q \in A)(q \leq_\mathbb{P} p)\} \in N$ is dense.]

23 Optional: Martin’s Maximum

A forcing $\mathbb{P}$ is called *stationary-preserving* if $\mathbb{P}$ does not destroy stationary subsets of $\omega_1$: if $\mathbb{M} \models (S$ is a stationary subset of $\omega_1)$, then $\mathbb{M}[G] \models (S$ is a stationary subset of $\omega_1)$, whenever $G$ is $\mathbb{P}$–generic over $\mathbb{M}$. *Martin’s Maximum* is the statement $MM$: for every stationary-preserving forcing $\mathbb{P}$, if $(\forall \alpha < \omega_1)(D_\alpha$ is dense open in $\mathbb{P})$, then there exists a $\{D_\alpha : \alpha < \omega_1\}$–generic set $G$ in $\mathbb{P}$.

(i) Show that if $\mathbb{P}$ has the countable chain condition, then $\mathbb{P}$ is stationary-preserving.
(ii) Give an example of a stationary-preserving forcing that has an uncountable anti-chain.

(iii) Prove $MM$ implies $MA_{\aleph_1}$.

REMARK. The relative consistency strength of $ZFC + MM$ is far stronger than that of $ZFC + MA$ which is equiconsistent with $ZFC$. $MM$ requires a large cardinal axiom for its consistency.

24 Optional: Normal Functions and Mahlo Cardinals

A (class) function $G : \text{Ord} \to \text{Ord}$ is called normal if $G$ is (strictly) increasing ($\alpha < \beta \to G(\alpha) < G(\beta)$) and continuous (for all limit $\delta \in \text{Ord}, G(\delta) = \bigcup_{\alpha < \delta} G(\alpha)$). A (strongly) inaccessible cardinal $\kappa$ is called (strongly) Mahlo if $\{ \alpha < \kappa : \alpha = \text{cf}(\alpha) \}$ is stationary in $\kappa$.

(i) (a) Prove in $ZFC$ that every normal function $G$ has a fixed point: there is $\delta \in \text{Ord}$ such that $G(\delta) = \delta$.

(b) Call the statement ”every normal function has a regular fixed point” the regular fixed point axiom $RFPA$. Show that $RFPA$ is not provable in $ZFC$.

(ii) (a) Suppose $\kappa$ is strongly Mahlo. Show $\{ \alpha < \kappa : \alpha$ is strongly inaccessible $\}$ is stationary in $\kappa$

(b) Prove that if $\kappa$ is strongly Mahlo, then $V_\kappa \models RFPA$.

25 Optional: Dilworth’s Theorem and Galvin’s Conjecture

(i) Prove Dilworth’s theorem: If a partial order $\mathbb{P}$ has at least $n^2 + 1$ elements for some $n < \omega$, then it has either a chain of size $n + 1$ or an anti-chain of size $n + 1$. Equivalently, if the largest antichain in $\mathbb{P}$ has size $n$, then $\mathbb{P}$ can be decomposed into $n$ chains.

(ii) Suggest and prove a generalization of Dilworth’s theorem for infinite cardinals. Give examples to support the alleged optimality of your result.

(iii) Galvin’s Conjecture

Let $\kappa_D$ be the least cardinal $\kappa$, if it exists, such that for every partial order $\mathbb{P}$, if every suborder of $\mathbb{P}$ of size less than $\kappa$ can be decomposed into countably many chains, then $\mathbb{P}$ can also be decomposed into countably many chains. Can you eliminate $\aleph_0$ and $\aleph_1$ as candidate values for $\kappa_D$? 

Galvin’s Conjecture states that $\aleph_2$ is a possible value for $\kappa_D$. See S. Todorcevic, Combinatorial dichotomies in set theory, Bull. Symbolic Logic, 17 (2011), 1-72.

26 Open Question: Banach Spaces, Cofinality and the Separable Quotient Problem

The cofinality of a Banach space $E$ is the least ordinal $\xi$ such that there exists an increasing chain $\{E_\alpha : \alpha < \xi\}$ of proper closed subspaces of $E$ whose union is dense in $E$. Does every infinite-dimensional Banach space have cofinality $\omega$?
REMARK. This is an equivalent reformulation of the Separable Quotient Problem: does every infinite-dimensional Banach space $X$ have a separable infinite-dimensional quotient $X/Y$? See S. Todorcevic, Combinatorial dichotomies in set theory, Bull. Symbolic Logic, 17 (2011), 1-72.

REMARK. After the classic works of Gödel and Cohen, the following are accessible and list many further suggestions for reading and research.


Dehornoy, P., Recent progress on the Continuum Hypothesis (after Woodin);

Koellner, P., The Continuum Hypothesis, Stanford Encyclopaedia of Philosophy, September 2011;
http://www.logic.harvard.edu/EFI_CH.pdf

Stephens, J., History of the Continuum in the Twentieth Century, to appear, Vol. 6 History of Logic;
http://www.math.yorku.ca/~steprans/Research/PDFSOfArticles/hoc2INDEXED.pdf

REMARK. For research problems in set theory, go to the sources; there are some treasure houses.


http://shelah.logic.at/files/702.pdf

Fremlin, D.H., Problems;
https://www.essex.ac.uk/maths/people/fremlin/problems.pdf

Miller, A.W., Some interesting problems;