

## TOPICS IN ANALYSIS (Lent 2025): Example Sheet 4

Comments, corrections are welcome at any time.

a.g.kovalev@dpmms.cam.ac.uk.

1. Prove that  $\sqrt{3} + \sqrt{5}$  and  $e^2$  are irrational.
2. (i) Show that a real number  $\alpha$  is algebraic if and only if it is a zero of a polynomial with rational coefficients.  
(ii) Is it true that if all the roots of a polynomial are algebraic, then the polynomial must have rational coefficients? Give a proof or counterexample.
3. By considering

$$\sum_{n=0}^{\infty} \frac{b_n}{10^{n!}}$$

with  $b_n \in \{1, 2\}$ , give another proof that the set of transcendental numbers is uncountable.

4. (i) We work in  $\mathbb{C}$ . Show that there exists a sequence of polynomials  $P_n$  such that

$$P_n(z) \rightarrow \begin{cases} 1 & \text{if } |z| \leq 1 \text{ and } \operatorname{Re} z \geq 0 \\ 0 & \text{if } |z| \leq 1 \text{ and } \operatorname{Re} z < 0 \end{cases}$$

as  $n \rightarrow \infty$ .

[Hint: If  $\Omega_1$  and  $\Omega_2$  are disjoint open sets and  $f(z) = 0$  for  $z \in \Omega_1$  and  $f(z) = 1$  for  $z \in \Omega_2$ , then  $f$  is holomorphic on  $\Omega_1 \cup \Omega_2$ .]

- (ii) Show that there exists a sequence of polynomials  $Q_n$  such that

$$P_n(z) \rightarrow \begin{cases} 1 & \text{if } \operatorname{Re} z \geq 0 \\ 0 & \text{if } \operatorname{Re} z < 0 \end{cases}$$

as  $n \rightarrow \infty$ .

5. Find the continued fraction expansions for  $100/37$  and  $\sqrt{7}$ .
6. Which rational with denominator less than 10 best approximates  $71/49$  and why?
7. Let  $a, b$  be positive integers. Define

$$x = \frac{1}{a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \dots}}}}$$

with the  $a$ 's and  $b$ 's continuing to alternate.

- (a) Show that  $x$  solves  $ax^2 + abx - b = 0$ .

(b) Now let  $a = b = 1$ . Show that, in this case,  $x = (-1 + \sqrt{5})/2$  and that if  $p_n/q_n$  is the  $n$ th convergent of the continued fraction above, then  $p_n = F_n$ ,  $q_n = F_{n+1}$ , where  $F_0, F_1, F_2, \dots$  are the Fibonacci numbers defined by  $F_0 = 0$ ,  $F_1 = 1$  and  $F_{n+1} = F_n + F_{n-1}$ .

(c) Show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}.$$

<sup>+</sup>(d) Prove the identities  $F_{2n+1} = F_n^2 + F_{n+1}^2$  and  $F_{2n} = F_n(F_{n-1} + F_{n+1})$ . Denote by  $x_n$  the ratio  $F_{n+1}/F_n$ . Use the above identities to express  $x_{2n}$  as a rational function of  $x_n$ . Let  $y_k = x_{2^k}$ . Explain why the sequence  $(y_k)$  converges very rapidly to the golden ratio  $(1 + \sqrt{5})/2$ .

**8.** Observe that the continued fraction obtained in the lectures for  $\tan x$ ,  $x \in [-1, 1]$ , remains valid in the more general case when  $x = z \in \mathbb{C}$  and  $|z| \leq 1$ .

Hence obtain an expansion for  $\tanh y$  with  $y$  real. Deduce that

$$\frac{e+1}{e-1} = 2 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \dots}}}}$$

Why does this give another proof that  $e$  is irrational?

**9.** Suppose that  $f : [1, \infty) \rightarrow \mathbb{R}$  is a continuous function such that  $f(nx) \rightarrow 0$  as  $n \rightarrow \infty$  for each fixed  $x$ . Show that  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ . [Hint: For  $\varepsilon > 0$ , consider the sets  $Q_k = \{x \in [1, \infty) : |f(nx)| \leq \varepsilon \forall n \geq k\}$ .]

**10.** Let  $A_j$  be a sequence of subsets of  $[0, 1]$  such that for each  $N \geq 1$ ,  $\cup_{j=N}^{\infty} A_j$  is open and dense in  $[0, 1]$ . Prove that the set  $S$  of points  $x \in [0, 1]$  such that  $x \in A_j$  for infinitely many  $j$  is dense. Must  $S$  be open? Must it be true that  $\cap_{j=1}^{\infty} A_j \neq \emptyset$ ?

**11.** If  $G$  is an open dense subset of  $\mathbb{R}$ , and  $\mathbb{Q}$  is the set of rationals, show that  $G \setminus \mathbb{Q}$  must be dense in  $\mathbb{R}$ . If we only assume  $G$  is uncountable and dense in  $\mathbb{R}$ , does it still follow that  $G \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$ ?

**12.** Show that there is a dense set of real numbers  $x$  with the property that for each positive integer  $n$ , there exist integers  $p$  and  $q$  with  $q \geq 2$  such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$