Proofs for some results in Topics in Analysis

T. W. Körner

October 4, 2022

Small print These proofs are offered on an 'as is' basis. I will not necessarily give the same proofs or use the same notation in the lectures. I very much hope that students will only consult these notes if they cannot provide their own proofs. These notes are intended for home use (if at all) and not as a means of following the live lectures.

There are certainly many small errors and quite likely some big ones. I should **very much** appreciate being told of any corrections or possible improvements to these notes.

These notes are written in LaTeX 2_ε and should be available in tex and pdf format from my home page

http://www.dpmms.cam.ac.uk/~twk/

Solution for Exercise 1.6. (i) If $y \in B(x,r)$ then $\delta = r - d(x,y) > 0$. Now observe that, if $z \in B(y,\delta)$, then

$$d(x,z) \leq d(x,y) + d(y,z) < d(y,x) + \delta < r.$$

(ii) If $y_n \in \bar{B}(x,r)$ and $y_n \underset{d}{\to} y$, then

$$d(x,y) \leq d(x,y_n) + d(y_n,y) \leq r + d(y_n,y) \to r,$$

so $d(x,y) \leq r$ and $y \in \bar{B}(x,r)$.

(iii) Suppose that $X \setminus E$ is not open. Then there is a point $y \notin E$ such that $B(y,r) \cap E \neq \varnothing$ whenever $r > 0$. Choose $y_n \in B(y,1/n) \cap E$. We have $y_n \in E$, $y_n \underset{d}{\to} y$ and yet $y \notin E$. Thus E is not closed.

(iii) Suppose that $X \setminus E$ is not closed. Then there is a sequence $y_n \notin E$ with $y_n \underset{d}{\to} y$ and yet $y \in E$. Thus $B(y,r) \nsubseteq E$ for all $r > 0$ and E is not open. □

Solution for Exercise 1.10. Suppose $x_n \underset{d}{\to} x$. Let $\epsilon > 0$. We can find an N such that $d(x_n,x) < \epsilon/2$ for all $n \geq N$. It follows that

$$d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all n, $m \geq N$. □

Proof of Lemma 1.11. (i) Let $\epsilon > 0$. We can find an N such that $d(x_n,x_m) < \epsilon/2$ for m, $n \geq N$. We can now find a J such that $n(J) \geq N$ and $d(x_{n(J)},x) < \epsilon/2$. We now observe that, if $m \geq N$, we get

$$d(x_m,x) \leq d(x_m,x_{n(J)}) + d(x_{n(J)},x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

(ii) If x_n is Cauchy, we can find a strictly increasing sequence $n(j)$ with

$$d(x_n,x_m) < \epsilon(j)$$

for all n, $m \geq n(j)$. By hypothesis, $x_{n(j)}$ converges as $j \to \infty$. Part (i) now tells us that the sequence x_n converges. □

Solution to Exercise 1.12. (i) Observe that, whenever x, y, $z \in Y$,

$$\begin{gathered}
d_Y(x,y) = d(x,y) \geq 0, \\
d_Y(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y, \\
d_Y(x,y) = d(x,y) = d(y,x) = d_Y(y,x), \\
d_Y(x,y) + d_Y(y,z) = d(x,y) + d(y,z) \geq d(x,z) = d_Y(x,z).
\end{gathered}$$

(ii) Suppose the sequence x_n is Cauchy in (Y, d_Y). Then the sequence x_n is Cauchy in (X, d), so $x_n \to x$ for some $x \in X$. But Y is closed, so $x \in Y$ and $x_n \to x$ in (Y, d_Y).

(iii) If $y_n \in Y$ and $y_n \to y$ in (X, d), then y_n is Cauchy in (X, d), so Cauchy in (Y, d_Y), so $y_n \to z$ in (Y, d_y) for some $z \in Y$. It follows that $y_n \to z$ in (X, d) so, by the uniqueness of limits, $y = z \in Y$. Thus Y is closed. □

Proof of Theorem 1.13 for $n = 2$. We prove the case when $n = 2$. Suppose that $\mathbf{x}_n = (x_n, y_n)$ is Cauchy in $\mathbb{R}^2$. Since

$$|x_n - x_m| \leq \|\mathbf{x}_n - \mathbf{x}_m\|,$$

x_n is Cauchy in $\mathbb{R}$, and by our 1A theorem (Theorem 1.7) converges to a limit x. Similarly y_n converges to a limit y in $\mathbb{R}$. If we set $\mathbf{x} = (x, y)$, then

$$\|\mathbf{x}_n - \mathbf{x}\| \leq |x_n - x| + |y_n - y| \to 0$$

as $n \to \infty$. □

Proof of Theorem 2.1. We prove the case $m = 2$. Write $\mathbf{x}_n = (x_n, y_n)$. We have that x_n is a bounded sequence in $\mathbb{R}$ and so (by the 1A result) there exists an $x \in \mathbb{R}$ and a sequence $n(j) \to \infty$ such that $x_{n(j)} \to x$ as $j \to \infty$. Now $y_{n(j)}$ is a bounded sequence in $\mathbb{R}$ and so there exists a $y \in \mathbb{R}$ and a sequence $j(k) \to \infty$ such that $y_{n(j(k))} \to y$ as $k \to \infty$. Now set $r(k) = n(j(k))$ and $\mathbf{x} = (x, y)$ to obtain $r(k) \to \infty$ and $\mathbf{x}_{r(k)} \to \mathbf{x}$ as $k \to \infty$. □

Theorem 2.2. (i) Since $\mathbf{x}_r \in E$, we know that the $\mathbf{x}_r$ form a bounded sequence and so have a convergent subsequence $\mathbf{x}_{r(k)} \to \mathbf{x}$. Since E is closed, $\mathbf{x} \in E$.

(ii) If E is not bounded, we can find $\mathbf{x}_r \in E$ with $\|\mathbf{x}_{r+1}\| \geq \|\mathbf{x}_r\| + 1$. If $r > s$

$$\|\mathbf{x}_r - \mathbf{x}_s\| \geq \|\mathbf{x}_r\| - \|\mathbf{x}_s\| \geq 1,$$

so no subsequence can be Cauchy and so no subsequence can converge.

If E is not closed, we can find $\mathbf{x}_r \in E$ and $\mathbf{x} \notin E$ such that $\mathbf{x}_r \to \mathbf{x}$. Any subsequence of $\mathbf{x}_r$ will still converge to $\mathbf{x} \notin E$. □

Solution to Exercise 2.4. (i) Suppose $f^{-1}(U)$ is open whenever U is. If $x \in X$, $\epsilon > 0$, we know that $B(f(x), \epsilon)$ is an open subset of Y, so $f^{-1}\big(B(f(x), \epsilon)\big)$ is an open subset of X containing x. Thus we can find a $\delta > 0$ with $B(x, \delta) \subseteq f^{-1}\big(B(f(x), \epsilon)\big)$. In other words,

$$z \in B(x, \delta) \Rightarrow f(z) \in B(f(x), \epsilon).$$

Thus f is continuous.

Conversely, if f is continuous and U open in Y, then, given $x \in X$ with $f(x) \in U$, we can find a $\delta > 0$ such that $B(f(x), \delta) \subseteq U$ and an $\epsilon > 0$ such that

$$z \in B(x, \delta) \Rightarrow f(z) \in B(f(x), \epsilon).$$

Thus $B(x, \epsilon) \subseteq f^{-1}(U)$. We have shown that $f^{-1}(U)$ is open.

(ii) Complementation. If $f^{-1}(F)$ is closed for all F closed then

$$U \text{ open} \Rightarrow Y \setminus U \text{ closed} \Rightarrow X \setminus f^{-1}(U) = f^{-1}(Y \setminus U) \text{ closed} \Rightarrow f^{-1}(U) \text{ open},$$

so f is continuous.

The converse is proved similarly. □

Proof of Lemma 2.5. If $d(x, A) = 0$, then we can find $x_n \in A$ such that $d(x_n, x) \leq 1/n$, so $x_n \to x$. But A is closed, so $x \in A$.

Let $x, y \in X$. Given $\epsilon > 0$, we can find $a \in A$ such that $d(x, a) \leq d(x, A) + \epsilon$. Now

$$d(y, A) \leq d(y, a) \leq d(x, y) + d(x, a) \leq d(x, y) + d(x, A) + \epsilon.$$

Since ϵ was arbitrary,

$$d(y, A) \leq d(x, y) + d(x, A).$$

The same argument shows that $d(x, A) \leq d(x, y) + d(y, A)$ so

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

This shows that the map $x \mapsto d(x, A)$ is continuous. □

Proof of Theorem 2.6. Suppose that $y_n \in f(E)$. Then $y_n = f(x_n)$ for some $x_n \in E$. By the Bolzano–Weierstrass property, we can find $n(j) \to \infty$ and $x \in E$ such that $x_{n(j)} \to x$ as $j \to \infty$. Now, by continuity,

$$y_{n(j)} = f(x_{n(j)}) \to f(x) \in f(E)$$

so we are done. □

Proof of Theorem 2.7. By Theorem 2.6, $f(E)$ is closed and bounded. Since $f(E)$ is non-empty, it has a supremum (see 1A), α, say. By the definition of the supremum, we can find $\mathbf{a}_n \in E$ such that

$$\alpha - 1/n \leq f(\mathbf{a}_n) \leq \alpha.$$

By the Bolzano–Weierstrass property, we can find $n(j) \to \infty$ and $\mathbf{a} \in E$ such that $\mathbf{a}_{n(j)} \to \mathbf{a}$ as $j \to \infty$. We have $f(\mathbf{a}_{n(j)}) \to f(\mathbf{a})$, so $f(\mathbf{a}) = \alpha$. Thus

$$f(\mathbf{a}) \geq f(\mathbf{x})$$

for all $\mathbf{x} \in E$. We find $\mathbf{b}$ in a similar manner. □

Proof of Theorem 2.9. Let $n \geq 1$. Suppose $P(z) = \sum_{j=0}^{n} a_j z^j$ where, without loss of generality, we take $a_n = 1$.

If $R \geq 2(2 + \sum_{j=0}^{n-1} |a_j|)$, then, whenever $|z| \geq R$, we have

$$\begin{aligned} |P(z)| &\geq |z|^n - \sum_{j=0}^{n-1} |a_j||z|^j \\ &= |z|^n \left(1 - \sum_{j=0}^{n-1} |a_j||z|^{j-n}\right) \\ &\geq |z|^n/2 > |a_0|. \end{aligned}$$

Since $\bar{D}_R = \{z \in \mathbb{C} : |z| \leq R\}$ is closed and bounded (that is to say compact) and the map $z \mapsto |P(z)|$ is continuous, $|P|$ attains a minimum on $\bar{D}_R$ at a point z_0, say. By the previous paragraph, $|z_0| < R$ (since $|P(z_0)| \leq |P(0)|$) and so we can find a $\delta > 0$ such that $|P(z)| \geq |P(z_0)|$ for all $|z - z_0| < \delta$.

By replacing $P(z)$ by $P(z - z_0)$, we may assume that $z_0 = 0$ so that $|P(z)| \geq |P(0)|$ for all $|z| < \delta$. If $a_0 = 0$, then we have $P(0) = 0$ and we are done.

We show that the assumption that $a_0 \neq 0$ leads to a contradiction. Observe that

$$P(z) = \sum_{j=m}^{n} a_j z^j + a_0 = a_0 \left(1 - \sum_{j=m}^{n} b_j z^j\right)$$

with $a_m \neq 0$ and so $b_m \neq 0$. Choose θ so that $b_m \exp(im\theta)$ is real and positive. Then

$$|P(\eta \exp i\theta)| \leq |a_0| - |b_m|\eta^m + |a_0|\eta^{m+1} \sum_{j=m}^{n} |b_j| \leq |a_0| - |b_m|\eta^m/2 < |P(0)|$$

when η is strictly positive and sufficiently small. We have the required contradiction. □

Solution to Exercise 2.10. (i) Let $S(m)$ be the statement that, if P is a polynomial of degree n with $n \leq m$ and $a \in \mathbb{C}$, then there exists a polynomial Q of degree $n - 1$ and an $r \in \mathbb{C}$ such that

$$P(z) = (z - a)Q(z) + r.$$

Suppose that $S(m)$ is true and P is a polynomial of degree $m + 1$. Then $P(z) = Az^{m+1} + Q(z)$ where $A \neq 0$ and Q is a polynomial of degree at most m. We have

$$P(z) = A(z - a)z^m + q(z)$$

where $q(z) = Q(z) + az^m$, so q is a polynomial of degree at most m and, by the inductive hypothesis,

$$q(z) = (z - a)u(z) + r$$

with u a polynomial of degree at most $m - 1$. Thus $P(z) = (z - a)Q(z) + r$ with $Q(z) = Az^m + u(z)$. We have shown that $S(m + 1)$ is true.

Now $S(1)$ is true, since $cz + d = c(z - a) + (d - ca)$, so the required result follows by induction.

(ii) We have $P(z) = (z - a)Q(z) + r$ by (i). Setting $z = a$, we have $0 = P(a) = r$ so $r = 0$ and the result follows.

(iii) If P_n has degree $n \geq 1$, then the fundamental theorem of algebra tells us that P_n has a root a_n. By (ii), there exists a polynomial P_{n-1} of degree $n - 1$ such that

$$P(z) = (z - a_n)P_{n-1}(z).$$

Using induction, we deduce that

$$P_n(z) = P_0(z) \prod_{j=1}^{n} (z - a_j),$$

where $P_0(z)$ is a polynomial of degree 0, that is to say, $P_0(z) = A$ with A a constant.

(iv) If P is not the zero polynomial, then (iii) tells we can find $m \leq n$ such that

$$P(z) = A \prod_{j=1}^{m} (z - a_j)$$

with A, a_1, a_2, ..., $a_m \in \mathbb{C}$ and $A \neq 0$. Now $P(z) = 0$ if and only if $z = a_j$ for some $1 \leq j \leq m$. The result follows. □

Solution to Exercise 3.2. There are a wide variety of ways of doing this exercise. Any way that works is fine.

(i) If $x \in \operatorname{Int} E$, we can find a $\delta > 0$ such that $B(x, 2\delta) \subseteq E$. If $y \in B(x, \delta)$, then, by the triangle inequality,

$$z \in B(y, \delta) \Rightarrow z \in B(x, 2\delta) \subseteq E.$$

Thus $\operatorname{Int} E$ is open.

If V is open and $V \subseteq E$, then, if $v \in V$, there exists a $\delta > 0$ with $B(v, \delta) \subseteq V \subseteq E$. Thus $V \subseteq \operatorname{Int} E$.

(ii) If $x_n \in \operatorname{Cl} E$ and $x_n \to x$, then we can find $y_n \in E$ such that $d(y_n, x_n) < 1/n$. By the triangle inequality,

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) \to 0 + 0 = 0$$

so $x \in \operatorname{Cl} E$.

If F is closed and $F \supseteq E$, then, whenever $x_n \in E$ and $x_n \to x$, we have $x_n \in F$, so $x \in F$. Thus $F \supseteq \operatorname{Cl} E$.

(iii) The complement of an open set is closed and the intersection of two closed sets is closed, so

$$\partial E = \operatorname{Cl} E \cap (\operatorname{Int} E)^c$$

is closed.

(iv) If E is closed, then we can find an $R > 0$ such that $E \subseteq \bar{B}(0, R)$. Since $\bar{B}(0, R)$ is closed, $\operatorname{Cl} E \subseteq \bar{B}(0, R)$. □

Proof of Lemma 3.3. We prove the result for $m = 2$. Since $\operatorname{Cl} \Omega$ is compact, we know that ϕ attains a maximum at some point $(x_0, y_0) \in \operatorname{Cl} \Omega$. We need to show that it is impossible that $(x_0, y_0) \in \Omega$.

Suppose, if possible, that $(x_0, y_0) \in \Omega$. Since Ω is open, we can find a $\delta > 0$ such that $B\big((x_0, y_0), \delta\big) \subseteq \Omega$. Consider the function $f(y) = \phi(x_0, y)$ defined for $y \in (y_0 - \delta, y_0 + \delta)$. We have f twice differentiable with a maximum at y_0. Thus, by 1A analysis, $f''(y_0) \leq 0$. It follows that

$$\frac{\partial^2 \phi}{\partial y^2}(x_0, y_0) \leq 0.$$

The same argument applies for the partial derivatives with respect to x, so

$$\nabla^2 \phi(x_0, y_0) = \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0) \leq 0$$

contradicting our hypotheses. □

Proof of Theorem 3.4. Again we prove the result for $m = 2$. Let $\psi(x, y) = x^2 + y^2$. Since ψ is continuous and $\operatorname{Cl} \Omega$ is compact, we know that there exists a M with $M \geq \psi(x, y)$ for all $(x, y) \in \operatorname{Cl} \Omega$. By direct calculation, $\nabla^2 \psi = 4$ everywhere.

Set $\phi_n = \phi + n^{-1} \psi$. Then ϕ_n satisfies the conditions of Lemma 3.3 with $\epsilon = 4/n$. It follows that there is an $\mathbf{x}_n = (x_n, y_n) \in \partial\Omega$ with

$$\phi_n(\mathbf{x}_n) \geq \phi_n(\mathbf{t})$$

for all $\mathbf{t} \in \operatorname{Cl} \Omega$. Automatically,

$$\phi(\mathbf{x}_n) \geq \phi(\mathbf{t}) - 8M/n.$$

Since $\partial\Omega$ is compact, we can find an $\mathbf{x} \in \partial\Omega$ and $n(j) \to \infty$ such that $\mathbf{x}_{n(j)} \to \mathbf{x}$. By continuity

$$\phi(\mathbf{x}) \geq \phi(\mathbf{t})$$

for all $\mathbf{t} \in \operatorname{Cl} \Omega$. □

Solution to Exercise 3.5. The map $z \mapsto |f(z)|$ is continuous so, by compactness, there exists a $z_0 = x_0 + iy_0 \in \operatorname{Cl}\Omega$ with $|f(z_0)| \geq |f(z)|$ for all $z \in \operatorname{Cl}\Omega$. By replacing $f(z)$ by $e^{i\theta}f(z)$, we may assume that $f(z_0)$ is real and positive.

Write $f(x+iy) = u(x,y) + iv(x,y)$ with u and v real. We have

$$u(x_0, y_0) = |f(z_0)| \geq |f(x+iy)| \geq u(x,y)$$

and u satisfies Laplace's equation. Thus there exists a $x_1 + iy_1 = z_1 \in \partial\Omega$ such that $u(x_1, y_1) = u(x_0, y_0)$ and so $|f(z_1)| \geq |f(z)|$ for all $z \in \operatorname{Cl}\Omega$. □

Proof of Theorem 3.6. Observe that, if $\tau = \phi - \psi$, then τ satisfies the conditions of Theorem 3.4 and so attains its maximum on $\partial\Omega$. But $\tau = 0$ on $\partial\Omega$. Thus $\tau(\mathbf{x}) \leq 0$ for $\mathbf{x} = \operatorname{Cl}\Omega$. The same argument applied to $-\tau$ shows that $-\tau(\mathbf{x}) \leq 0$ for $\mathbf{x} = \operatorname{Cl}\Omega$. Thus $\tau = 0$ on $\operatorname{Cl}\Omega$ and we are done. □

Solution to Exercise 3.7. (i) If $\mathbf{x} \in \Omega$, then, setting

$$\delta = \min\{\|\mathbf{x}\|, 1 - \|\mathbf{x}\|\},$$

we have $\delta > 0$ and $B(\mathbf{x}, \delta) \subseteq \Omega$. Thus Ω is open.

Observe that $(0, 1/n) \to (0,0)$, so $\mathbf{0} \in \operatorname{Cl}\Omega$. Again, if $\|\mathbf{x}\| = 1$, then $(1 - 1/n)\mathbf{x} \to \mathbf{x}$, so $\mathbf{x} \in \operatorname{Cl}\Omega$. Thus $\operatorname{Cl}\Omega \supseteq \bar{B}(\mathbf{0}, 1)$. Since $\bar{B}(\mathbf{0}, 1)$ is closed $\operatorname{Cl}\Omega = \bar{B}(\mathbf{0}, 1)$.

Finally,

$$\partial\Omega = \operatorname{Cl}\Omega \setminus \operatorname{Int}\Omega = \operatorname{Cl}\Omega \setminus \Omega = \{\mathbf{x} \in \mathbb{R}^2 \,:\, \|\mathbf{x}\| = 1\} \cup \{\mathbf{0}\}.$$

(ii) Let T be a rotation with centre the origin. If $\psi = \phi T$, then (using the chain rule if you do not know the result already from applied courses)

$$\nabla^2\psi = 0.$$

But

$$\psi(\mathbf{x}) = \begin{cases} 0 & \text{if } \|\mathbf{x}\| = 1, \\ 1 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Thus, by uniqueness, $\psi = \phi$ and so, since T was an arbitrary rotation,

$$\phi(\mathbf{x}) = f(\|\mathbf{x}\|)$$

for some function $f : [0,1] \to \mathbb{R}$.

(iii) The chain rule gives

$$\frac{\partial\phi}{\partial x} = f'(r)\frac{x}{r} \text{ and } \frac{\partial^2\phi}{\partial x^2} = f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

so, using the parallel result for derivatives with respect to y,

$$\nabla^2\phi = f''(r) + f'(r)r^{-1} = r^{-1}\frac{d}{dr}\big(rf(r)\big).$$

(Or we can just quote this result from applied courses.) Thus

$$\frac{d}{dr}\big(rf(r)\big) = 0$$

so $rf'(r) = B$ and $f(r) = A + B\log r$ for appropriate constants A and B.

(iv) We need $f(r) \to 1$ as $r \to 0+$, so $B = 0$ and $A = 1$. This gives $f(1) = 1$, contradicting the condition $\phi(\mathbf{x}) = 0$ if $\|\mathbf{x}\| = 1$. $\square$

Proof of Lemma 4.4. Observe that $f = g^{-1}Fg$ is a continuous function from $\bar{D}$ to $\bar{D}$ and so, by Theorem 4.3, has a fixed point w. Set $a = g(w)$. $\square$

Proof of Theorem 4.5. (i)$\Rightarrow$ (ii) Suppose, if possible, that there exists a continuous function $g : \bar{D} \to \partial D$ with $g(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial D$. If T is a rotation through π about $\mathbf{0}$, then $f = T \circ g$ is a continuous function from $\bar{D}$ to itself with no fixed points, contradicting (i).
(ii)$\Rightarrow$ (i) Suppose, if possible, that $f : \bar{D} \to \bar{D}$ is a continuous function with no fixed points. If we define

$$E = \{(\mathbf{x}, \mathbf{y}) \in \bar{D}^2 \,:\, \mathbf{x} \neq \mathbf{y}\}$$

and $u : E \to \partial D$ by taking $u(\mathbf{x}, \mathbf{y})$ to be the point where the straight line joining $\mathbf{x}$ to $\mathbf{y}$ in the indicated direction cuts ∂D then u is continuous. (We shall take this as geometrically obvious. The algebraic details are messy (but made easier if you use the fact that the composition of continuous functions is continuous). The really conscientious student can do Exercise 18.14.) Using the chain rule for continuous functions, we see that

$$g(\mathbf{x}) = u\big(\mathbf{x}, f(\mathbf{x})\big)$$

defines a retraction mapping from $\bar{D}$ to ∂D, contradicting (ii). $\square$

Proof of Lemma 4.6. (i)$\Rightarrow$(ii) Suppose, if possible, that $\tilde{k}$ exists with the properties stated in (ii), Then, if T is a rotation through π, about $\mathbf{0}$, we see that $f = T \circ \tilde{k}$ is a continuous map from $\bar{D}$ to $\bar{D}$ without a fixed point. By Theorem 4.5 this contradicts (i).
(ii)$\Rightarrow$(i) If $\tilde{k}$ is a continuous retract from $\bar{D}$ to ∂D, then it certainly satisfies (ii).
(iii)$\Leftrightarrow$(ii) We use an argument of the type used for Lemma 4.4. $\square$

Proof of Lemma 4.7. (ii)$\Rightarrow$(i) Let $h : \bar{T} \to \partial T$ be continuous with $h(I) \subseteq I$, $h(J) \subseteq J$, $h(K) \subseteq K$. Let $A = h^{-1}(I)$, $B = h^{-1}(J)$, $C = h^{-1}(K)$. Since h is continuous A, B and C are closed. Since $I \cup J \cup K = \partial D$, $A \cup B \cup C = \bar{D}$. But

$$A \cap B \cap C = h^{-1}(I) \cap h^{-1}(J) \cap h^{-1}(K) = h^{-1}(I \cap J \cap K) = h^{-1}(\varnothing) = \varnothing$$

contradicting (ii).
(i)$\Rightarrow$(ii) Suppose that A, B and C are closed subsets of T with $A \supseteq I$, $B \supseteq J$, $C \supseteq K$, $A \cup B \cup C = T$, and $A \cap B \cap C = \varnothing$.

We consider T as the triangle

$$T = \{(x, y, z) \in \mathbb{R}^3 \,:\, x + y + z = 1,\ x,\, y,\, z \geq 0\}.$$

(In my school days we called these 'barycentric coordinates'.) If $\mathbf{x} \in T$, we know that $\mathbf{x}$ lies in at most two of the sets A, B and C so (by Lemma 2.5) at least one of $d(\mathbf{x}, A)$, $d(\mathbf{x}, B)$ and $d(\mathbf{x}, C)$ is non-zero. Thus

$$h(\mathbf{x}) = \frac{1}{d(\mathbf{x}, A) + d(\mathbf{x}, B) + d(\mathbf{x}, C)} \big(d(\mathbf{x}, A), d(\mathbf{x}, B), d(\mathbf{x}, C)\big)$$

defines a continuous function $h : T \to T$. If $\mathbf{x} \in I$, then $d(\mathbf{x}, A) = 0$ and so $h(\mathbf{x}) \in I$. Similarly $h(J) \subseteq J$ and $h(K) \subseteq K$ contradicting (i). $\square$

Proof of Theorem 4.8. Given an edge of the grid joining vertices u and v we assign a value $E(u, v)$ to the edge by a rule which ensures that, if u and v have the same colour, $E(u, v) = 0$, if u and v, have different colours X and Y, then $E(u, v) = \zeta(X, Y)$ with $\zeta(X, Y) = -\zeta(Y, X)$ and $\zeta(X, Y) = \pm 1$.

The table which follows gives an example.

colour u	colour v	$E(u, v)$
R	R	0
R	G	1
R	B	-1
G	R	-1
G	G	0
G	B	1
B	R	1
B	G	-1
B	B	0

If uvw is a grid triangle then, by inspection, the sum of the edge values (going round anticlockwise) is zero unless all of the vertices have different colours. By considering internal cancellation, the total sum of the edge values is the sum of the edge values going round the outer edge and this is non-zero. Thus one of the grid triangles must have all three vertices of different colours. $\square$

Proof of Lemma 4.7 (ii). Suppose that A, B and C are closed subsets of T with $A \supseteq I$, $B \supseteq J$ and $C \supseteq K$ and $A \cup B \cup C = T$.

Take a triangular grid formed by n equally spaced parallel lines for each of the three sides dividing T into a grid of congruent triangles. Colour the vertices red, blue or green so that all the red vertices lie in A, all the blue vertices lie in B and all the green vertices lie in C, making sure that the outside edges are coloured as required by Lemma 4.8.

Lemma 4.8 tells us that there is a grid triangle with vertex $\mathbf{a}_n$ red, so in A, vertex $\mathbf{b}_n \in B$ and $\mathbf{c}_n \in C$. By compactness, we can find $n(j) \to \infty$ and $\mathbf{x} \in T$ such that $\mathbf{a}_{n(j)} \to \mathbf{x}$ and so $\mathbf{b}_{n(j)} \to \mathbf{x}$, $\mathbf{c}_{n(j)} \to \mathbf{x}$. Since A, B and C are closed $\mathbf{x} \in A \cap B \cap C$, so $A \cap B \cap C \neq \varnothing$ $\square$

Solution for Exercise 4.11. T is a closed triangle in the appropriate plane. If $\mathbf{X} \in T$ and we write $\mathbf{y} = T\mathbf{x}$, then $y_i \geq 0$ and

$$\sum_{i=1}^{3} y_i = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_j = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}x_j = \sum_{j=1}^{3} x_j = 1,$$

so $\mathbf{y} \in T$. Thus T is a continuous map of T into itself and has a fixed point $\mathbf{e}$. We observe that $\mathbf{e}$ is an eigenvector lying in T with eigenvalue 1. $\square$

Proof of Theorem 5.1. Let $\tilde{E} = \{(p, 1-p, q, 1-q) : 0 \leq p, q \leq 1\}$. (Thus $\tilde{E}$ is a two dimensional square embedded in $\mathbb{R}^4$.)

Suppose $(\mathbf{p}, \mathbf{q}) \in \tilde{E}$. Write

$$u_1(\mathbf{p}, \mathbf{q}) = \max\{0, A(1, 0, \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\}.$$

Thus u_1 is Albert's expected gain if, instead of choosing $\mathbf{p}$ when Bertha chooses $\mathbf{q}$, he chooses $(1, 0)$ and Bertha maintains her choice *provided this is positive* and u_1 is zero otherwise. Similarly

$$u_2(\mathbf{p}, \mathbf{q}) = \max\{0, A(0, 1, \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\},$$

so u_2 is Albert's expected gain if, instead of choosing $\mathbf{p}$ when Bertha chooses $\mathbf{q}$, he chooses $(0, 1)$ and Bertha maintains her choice *provided this is positive* and u_2 is zero otherwise. In the same way, we take

$$v_1(\mathbf{p}, \mathbf{q}) = \max\{0, B(\mathbf{p}, 1, 0) - B(\mathbf{p}, \mathbf{q})\}.$$

and

$$v_2(\mathbf{p}, \mathbf{q}) = \max\{0, B(\mathbf{p}, 0, 1) - B(\mathbf{p}, \mathbf{q})\}.$$

Now define

$$g(\mathbf{p}, \mathbf{q}) = (\mathbf{p}', \mathbf{q}')$$

with

$$\mathbf{p}' = \frac{\mathbf{p} + \mathbf{u}(\mathbf{p}, \mathbf{q})}{1 + u_1(\mathbf{p}, \mathbf{q}) + u_2(\mathbf{p}, \mathbf{q})}$$

and

$$\mathbf{q}' = \frac{\mathbf{q} + \mathbf{v}(\mathbf{p}, \mathbf{q})}{1 + v_1(\mathbf{p}, \mathbf{q}) + v_2(\mathbf{p}, \mathbf{q})}.$$

We observe that g is a well defined continuous function from $\tilde{E}$ into itself and so has a fixed point $(\mathbf{p}^*, \mathbf{q}^*)$.

We claim that $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash stable point.

Suppose, if possible, that $A\big((r, 1-r), \mathbf{q}^*\big) > A\big((p^*, 1-p^*), \mathbf{q}^*\big)$. Without loss of generality, we may suppose that $r > p^*$ so that

$$A\big((1,0), \mathbf{q}^*\big) > A\big((p^*, 1-p^*), \mathbf{q}^*\big)$$

and

$$A\big((0,1), \mathbf{q}^*\big) < A\big((p^*, 1-p^*), \mathbf{q}^*\big).$$

Thus $u_1(\mathbf{p}^*, \mathbf{q}^*) > 0$ and $u_2(\mathbf{p}^*, \mathbf{q}^*) = 0$, whence $\mathbf{p}^* = (1,0)$ and $u_1(\mathbf{p}^*, \mathbf{q}^*) = 0$ which contradicts our earlier assertion.

We have shown that

$$A\big(\mathbf{p}^*, \mathbf{q}^*\big) \geq A\big((p, 1-p), \mathbf{q}^*\big)$$

for all $1 \geq p \geq 0$. The same argument shows that

$$B\big(\mathbf{p}^*, \mathbf{q}^*\big) \geq B\big(\mathbf{p}^*, (q, 1-q)\big)$$

for all $1 \geq q \geq 0$ so we are done. $\square$

Solution of Exercise 5.2. Suppose that A swerves with probability a and B with probability b. The value of the game to A is

$$V(a,b) = -ab + 10(1-a)b - 5a(1-b) - 100(1-a)(1-b).$$

If $0 < a < 1$,$0 < b < 1$

$$\frac{\partial V}{\partial a}(a,b) = 95 - 106b,$$

so by symmetry we have a Nash equilibrium point $(a,b) = (95/106, 95/106)$. However

$$V(a,0) = -5a - 100(1-a) = 95a - 100, V(a,1) = 10 - 11a$$

so, again using symmetry, $(1,0)$ and $(0,1)$ are also Nash equilibrium points. $\square$

Solution of Exercise 6.3. If $\mathbf{x}'$, $\mathbf{y}' \in E'$ and $0 \leq t \leq 1$, then

$$x'_j = a_j x_j + b_j, \ y'_j = a_j y_j + b_j$$

with $\mathbf{x}$, $\mathbf{y} \in E$ and so

$$tx'_j + (1-t)y'_j = a_j(tx_j + (1-t)y_j) + b_j$$

for all j. But $t\mathbf{x} + (1-t)\mathbf{y} \in E$, since E is convex, so $t\mathbf{x}' + (1-t)\mathbf{y}' \in E'$ and is convex.

We now recall Theorem 2.6 and observe that E' is the continuous image of a compact set so compact. □

Proof of Lemma 6.4. If $\mathbf{x} \in K$ and $0 \leq t \leq 1$, then, since $\mathbf{1} \in K$ and K is convex, we have

$$(1-t)\mathbf{1} + t\mathbf{x} \in K$$

so, by our hypothesis,

$$\begin{aligned} 1 \geq \prod_{j=1}^{n} \big(tx_j + (1-t)\big) &= \prod_{j=1}^{n} \big(1 + t(x_j - 1)\big) \\ &= 1 + t\sum_{j=1}^{n}(x_j - 1) + t^2 P(t) \end{aligned}$$

where P is a polynomial with coefficients depending on $\mathbf{x}$. It follows that, if $0 \leq t \leq 1$, we have

$$0 \geq \sum_{j=1}^{n}(x_j - 1) + tP(t).$$

Allowing $t \to 0+$ gives

$$0 \geq \sum_{j=1}^{n}(x_j - 1),$$

which is the desired result. □

Proof of Theorem 6.5. The Nash conditions mean that the problem is invariant under affine transformation (i.e. transformations of the type discussed in Exercise 6.3). Thus we may assume that $\mathbf{s} = \mathbf{0}$. If the hyperboloid $\prod_{j=1}^{n} y_j = K$ touches the convex set E' at $\mathbf{y}$ (with $y_j > 0$) then the transformation $x_j = K^{-1/n} y_j / y_j^*$ gives a hyperboloid $\prod_{j=1}^{n} x_j = 1$ touching a convex set E at $(1, 1, \ldots, 1)$.

Thus we may assume that $\mathbf{s} = \mathbf{0}$ and $x_1^* = x_2^* = \cdots = x_n^* = 1$.

By Lemma 6.4, we have

$$K \subseteq L = \left\{ \mathbf{x} : \sum_{j=1}^{n} x_j \leq n \right\},$$

and, by the independence of irrelevant alternatives, if $\mathbf{x}^*$ is best for L, it is best for K. Now L is symmetric so any best point $\mathbf{x}$ for L must lie on $x_1 = x_2 = \ldots = x_n$. But, amongst these points, only $\mathbf{x}^*$ is Pareto optimal so we are done. □

Proof of Lemma 6.6. By compactness, there is a point $\mathbf{x}^*$ where f attains its maximum. By translation, we may suppose $\mathbf{s} = \mathbf{0}$ and, re-scaling the axes, we may suppose $\mathbf{x}^* = \mathbf{e} = (1, 1, \ldots, 1)$.

Lemma 6.4 tells us that

$$\{\mathbf{k} \in K \,:\, k_j \geq 0 \ \forall j\} \subseteq \{\mathbf{x} \in K \,:\, x_j \geq 0 \ \forall j \text{ and } x_1 + x_2 + \ldots + x_n = n\}.$$

The uniqueness of the maximum now follows from the conditions for equality in the arithmetic geometric inequality. □

Solution for Exercise 7.1. (i) We use induction on n to show that E is n times differentiable with

$$E^{(n)}(t) = P_n(1/t)E(t)$$

for all $t \neq 0$ and some polynomial P_n.

The result is certainly true for $n = 0$ with $P_0 = 1$. If it is true for $n = m$, then the standard rules for differentiation show that $E^{(m)}$ is differentiable with

$$E^{(m+1)}(t) = t^{-2}P_m'(1/t)E(t) - 2t^{-3}P_m(1/t)E(t) = P_{m+1}(1/t)E(t)$$

for all $t \neq 0$ and the polynomial $P_{m+1}(s) = s^2P_m'(s) - 2s^3P_m(s)$.

(ii) We use induction on n to show that E is n times differentiable at 0 with

$$E^{(n)}(0) = 0.$$

The result is true for $n = 0$. If it is true for $n = m$, then

$$\frac{E^{(m)}(h) - E^{(m)}(0)}{h} = h^{-1}P(h^{-1})E(h) \to 0$$

as $h \to 0$, so it is true for $n = m + 1$.

(iii) We have

$$E(t) \neq 0 = \sum_{n=0}^{\infty} \frac{0}{n!} t^n = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} t^n$$

for all $t \neq 0$, as stated. $\square$

Proof of Lemma 7.2. (i) If P and Q have degree at most n and

$$P(x_j) = Q(x_j) = f(x_j)$$

for $0 \leq j \leq n$, then $P - Q$ is a polynomial of degree at most n vanishing at at least $n + 1$ points. Thus $P - Q = 0$, by Exercise 2.10, so $P = Q$.

(ii) We observe that $e_j(x_i) = 1$ if $i = j$, but $e_j(x_i) = 0$ otherwise and that e_j is a polynomial of degree n. Thus

$$P = \sum_{j=0}^{n} f(x_j) e_j$$

is a polynomial of degree at most n with

$$P(x_i) = \sum_{j=0}^{n} f(x_j) e_j(x_i) = f(x_i)$$

for $0 \leq i \leq n$.

(iii) It is easy to check that $\mathcal{P}_n$ is a vector space. Part (ii) shows that the e_j span $\mathcal{P}_n$. If

$$\sum_{j=0}^{n} \lambda_j e_j = 0,$$

then

$$\lambda_i = \sum_{j=0}^{n} \lambda_j e_j(x_i) = 0$$

for each i, so the the e_j are linearly independent. $\square$

Proof of Theorem 7.3. By de Moivre's theorem,

$$\begin{aligned}
\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n\\
&= \sum_{r=0}^{n} i^r \binom{n}{r} (\cos\theta)^{n-r}(\sin\theta)^r\\
&= \sum_{0\le 2r\le n} (-1)^r \binom{n}{2r} (\cos\theta)^{n-2r}(\sin\theta)^{2r}\\
&\qquad + i\sin\theta \sum_{0\le 2r\le n-1} (-1)^r \binom{n}{2r+1} (\cos\theta)^{n-1-2r}(\sin\theta)^{2r}\\
&= \sum_{0\le 2r\le n} (-1)^r \binom{n}{2r} (\cos\theta)^{n-2r}\big(1-(\cos\theta)^2\big)^r\\
&\qquad + i\sin\theta \sum_{0\le 2r\le n-1} (-1)^r \binom{n}{2r+1} (\cos\theta)^{n-1-2r}\big(1-(\cos\theta)^2\big)^r\\
&= T_n(\cos\theta) + i\sin\theta U_{n-1}(\cos\theta),
\end{aligned}$$

where T_n is a polynomial of degree at most n and U_{n-1} a polynomial of degree at most $n-1$.

Taking real and imaginary parts, we obtain

$$T_n(\cos\theta) = \cos n\theta$$

for all θ and

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}$$

for $\sin\theta \neq 0$, $U_{n-1}(1) = n$, $U_{n-1}(-1) = (-1)^{n-1}n$. The roots of U_{n-1} and T_n can be read off directly and show that the two polynomials have full degree.

The coefficient of t^n in T_n is

$$\sum_{0\le 2r\le n} \binom{n}{2r} = \frac{1}{2}\big((1+1)^n + (1-1)^n\big) = 2^{n-1}$$

for $n \geq 1$. □

Solution of Exercise 7.4. The key result that we use in (i) and (ii) is that, if $f \in C([0,1])$, $f(t) \geq 0$ for all $t \in [0,1]$ and $\int_0^1 f(t)\,dt = 0$, then $f(t) = 0$ for all $t \in [0,1]$.

(i) Observe that

$$\|f\|_1 = \int_0^1 |f(t)|\,dt \geq 0$$

and that, if $\|f\|_1 = 0$, then

$$\int_0^1 |f(t)|\,dt = 0,$$

so $|f(t)| = 0$ for all t, so $f(t) = 0$ for all t and $f = 0$.

Further

$$\|\lambda f\|_1 = \int_0^1 |\lambda||f(t)|\,dt = |\lambda| \int_0^1 |f(t)|\,dt = |\lambda|\|f\|_1$$

and, since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$, we have

$$\|f + g\|_1 = \int_0^1 |f(t) + g(t)|\,dt \leq \int_0^1 |f(t)| + |g(t)|\,dt = \|f\|_1 + \|g\|_1,$$

so we have a norm.

(ii) We have

$$\langle f, f\rangle = \int_0^1 f(t)^2\,dt \geq 0.$$

If $\langle f, f\rangle = 0$, then $\int_0^1 f(t)^2\,dt = 0$, so $f(t)^2 = 0$ for all t, so $f(t) = 0$ for all t and $f = 0$.

We have

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 g(t)f(t)\,dt = \langle g, f\rangle$$

and

$$\begin{aligned}\langle f + g, h\rangle &= \int_0^1 (f(t) + g(t))h(t)\,dt \\ &= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt = \langle f, h\rangle + \langle g, h\rangle\end{aligned}$$

whilst

$$\langle \lambda f, g\rangle = \int_0^1 \lambda f(t)g(t)\,dt = \lambda \int_0^1 f(t)g(t)\,dt = \lambda\langle f, g\rangle,$$

so we have an inner product.

(iii) Observe that $|f(t)| \geq 0$,so

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| \geq 0,$$

that

$$\|f\|_\infty = 0 \Rightarrow \sup_{t\in[0,1]} |f(t)| = 0 \Rightarrow |f(t)| = 0\ \forall t \Rightarrow f = 0,$$

that

$$\|\lambda f\|_\infty = \sup_{t\in[0,1]} |\lambda f(t)| = \sup_{t\in[0,1]} |\lambda||f(t)| = |\lambda| \sup_{t\in[0,1]} |f(t)| = \lambda\|f\|_\infty,$$

and, that, since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$,

$$\begin{aligned}\|f+g\|_\infty &= \sup_{t\in[0,1]} |f(t)+g(t)| \leq \sup_{t\in[0,1]} (|f(t)| + |g(t)|)\\ &\leq \sup_{t,s\in[0,1]} (|f(t)| + |g(s)|) = \|f\|_\infty + \|g\|_\infty,\end{aligned}$$

so we are done.

The Cauchy–Schwarz inequality gives

$$\|f\|_2 = \|f\|_2\|1\|_2 \geq \langle |f|, 1\rangle = \|f\|_1.$$

First year analysis gives

$$\|f\|_\infty = \|f^2\|_\infty^{1/2} \geq \left(\int_0^1 f(t)^2\,dt\right)^{1/2} = \|f\|_2.$$

.

If f_n is as stated, $\|f_n\|_1 = 2\int_0^{1/n} nt\,dt = 1/n$, $\|f_n\|_\infty = 1$ and

$$\|f_n\|_2 = \left(2\int_0^{1/n} (nt)^2\,dt\right)^{1/2} = \left(\frac{2}{3n}\right)^{1/2}.$$

Thus $\|f_n\|_\infty/\|f_2\|_1 = (3/2)^{1/2}n^{1/2} \to \infty$ and $\|f_n\|_2/\|f_1\|_1 = (2/3)^{1/2}n^{1/2} \to \infty$ as $n \to \infty$. We have genuinely different measures of distance. □

Proof of Theorem 7.7. Suppose that f is not uniformly continuous. Then we can find an $\epsilon > 0$ and $\mathbf{x}_n$, $\mathbf{y}_n \in E$ such that

$$\|\mathbf{x}_n - \mathbf{y}_n\| \leq 1/n \text{ and } \|f(\mathbf{x}_n) - f(\mathbf{y}_n)\| \geq \epsilon.$$

By compactness, we can find $\mathbf{e} \in E$ and $n(j) \to \infty$ such that $\mathbf{x}_{n(j)} \to \mathbf{e}$. The triangle inequality tells us that $\mathbf{y}_{n(j)} \to \mathbf{e}$ and so

$$\|f(\mathbf{x}_{n(j)}) - f(\mathbf{y}_{n(j)})\| \leq \|f(\mathbf{x}_{n(j)}) - f(\mathbf{e})\| + \|f(\mathbf{y}_{n(j)}) - f(\mathbf{e})\| \to 0 + 0 = 0.$$

We have a contradiction. □

Proof of Theorem 7.8. By replacing X by $Y = X - \mathbb{E}X$, we may suppose that $\mathbb{E}X = 0$.

Let

$$\mathbb{I}_{\mathbb{R}\setminus(-a,a)}(t) = \begin{cases} 0 & \text{if } |t| < a, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\frac{t^2}{a^2} \geq \mathbb{I}_{\mathbb{R}\setminus(-a,a)}(t)$$

for all t, so, automatically,

$$\frac{X^2}{a^2} \geq \mathbb{I}_{\mathbb{R}\setminus(-a,a)}(X)$$

and

$$\frac{\sigma^2}{a^2} = \mathbb{E}\frac{X^2}{a^2} \geq \mathbb{E}\mathbb{I}_{\mathbb{R}\setminus(-a,a)}(X) = \Pr(|X| \geq a).$$

□

Proof of Theorem 7.9. (i) We have

$$\begin{aligned} p_n(t) &= \mathbb{E}f(Y_n(t)) \\ &= \sum_{j=0}^{n} f(j/n) \Pr(X_1 + X_2 + \cdots + X_n = j) \\ &= \sum_{j=0}^{n} \binom{n}{j} f(j/n) t^j (1-t)^{n-j}. \end{aligned}$$

(ii) Automatically,

$$\mathbb{E}Y_n = \mathbb{E}\frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{\mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_n}{n} = \frac{nt}{n} = t$$

and, since the X_j are independent,

$$\begin{aligned} \operatorname{var} Y_n &= \operatorname{var} \frac{X_1 + X_2 + \cdots + X_n}{n} = n^{-2} \operatorname{var}(X_1 + X_2 + \cdots + X_n) \\ &= n^{-2}(\operatorname{var} X_1 + \operatorname{var} X_2 + \ldots + \operatorname{var} X_n) \\ &= n^{-1} \operatorname{var} X_1 = n^{-1} t(1-t) \leq n^{-1}. \end{aligned}$$

Let $\epsilon > 0$. By uniform continuity we can find an $\eta > 0$ such that $|f(t) - f(s)| \leq \epsilon$ for $|t-s| \leq \eta$ and $t,\ s \in [0,1]$. Thus, using Chebychev's inequality,

$$\begin{aligned} |p_n(t) - f(t)| &= \big|\mathbb{E}\big(f(Y_n) - f(t)\big)\big| \leq \mathbb{E}|f(Y_n) - f(t)| \\ &\leq \epsilon \Pr(|Y_n - t| < \eta) + 2\|f\|_\infty \Pr(|Y_n - t| \geq \eta) \\ &\leq \epsilon + 2\|f\|_\infty \Pr(|Y_n - \mathbb{E}Y_n| \geq \eta) \\ &\leq \epsilon + 2\|f\|_\infty \eta^{-2}/n \leq 3\epsilon, \end{aligned}$$

provided only that $n \geq \epsilon^{-1}(2\|f\| + 1)\eta^{-2}$. Since ϵ is arbitrary, the result follows. $\square$

Proof of Theorem 8.1. Without loss of generality, suppose that

$$f(a_j) - P(a_j) = (-1)^j \sigma \text{ for all } 0 \leq j \leq n.$$

Suppose, if possible, that Q is a polynomial of degree $n-1$ or less such that $\|P - f\|_\infty > \|Q - f\|_\infty$.

We look at $R = P - Q$. Note first that R is a polynomial of degree at most $n-1$. If j is odd,

$$\begin{aligned} R(a_j) &= \big(P(a_j) - f(a_j)\big) + \big(f(a_j) - Q(a_j)\big) \\ &= |P(a_j) - f(a_j)| + \big(f(a_j) - Q(a_j)\big) \\ &\geq |P(a_j) - f(a_j)| - \|Q - f\|_\infty = \|P - f\|_\infty - \|Q - f\|_\infty > 0. \end{aligned}$$

and a similar argument shows that

$$R(a_j) < 0$$

when j is even.

The intermediate value theorem now tells that R has at least n zeros, so $R = 0$ and $P = Q$, contradicting our initial assumption. $\square$

Proof of Theorem 8.2. If $t = \cos\theta$, then

$$t^n - S_n(t) = 2^{1-n} T_n(t) = 2^{1-n} \cos n\theta$$

Thus

$$|t^n - S_n(t)| \leq 2^{1-n}$$

for $t \in [-1, 1]$ and

$$t^n - S_n(t) = (-1)^j 2^{1-n}$$

for $t = \cos j\pi/n$ $[0 \leq j \leq n]$.

The stated result now follows from the equiripple criterion. $\square$

Proof of Corollary 8.3. (i) This is just a restatement of Theorem 8.2.

(ii) Let $\Gamma(n)$ be the statement given in (ii) with the extra condition $\epsilon_n \leq 1$. $\Gamma(0)$ is true with $\epsilon_0 = 1$ by inspection.

Suppose that Γ_n is true, that $P(t) = \sum_{j=0}^{n+1} a_j t^j$ is a polynomial of degree at most $n+1$, and that $|a_k| \geq 1$ for some $n+1 \geq k \geq 0$. If $|a_{n+1}| \leq \epsilon_n/2$, then

$$P(t) = a_{n+1} t^{n+1} + Q(t)$$

where $Q(t) = \sum_{j=0}^{n} a_j t^j$ is a polynomial of degree at most $n+1$ and $|a_k| \geq 1$ for some $n \geq k \geq 0$. Thus

$$\|P\|_\infty \geq \|Q\|_\infty - |a_{n+1}| \geq \epsilon_n/2.$$

On the other hand, if $|a_{n+1}| \geq \epsilon_n/2$, then part (i) tells us that

$$\|P\|_\infty \geq 2^{-n+1}\epsilon_n/2 = 2^{-n}\epsilon_n.$$

Thus, whatever the value of a_{n+1},

$$\|P\|_\infty \geq 2^{-n-1}\epsilon_n$$

and $\Gamma(n+1)$ holds with $\epsilon_{n+1} = 2^{-n-1}\epsilon_n$.

The required result holds by induction. □

Proof of Theorem 8.4. By rescaling and translation, we may suppose that $[a, b] = [-1, 1]$. Consider the map $F : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $F(\mathbf{a}) = \|f - Q\|_\infty$ where

$$Q(t) = \sum_{j=0}^{n} a_j t^j.$$

Recalling the inequality $\big||d(f,g)| - |d(f,h)|\big| \leq d(g,h)$, we have

$$|F(\mathbf{a}) - F(\mathbf{b})| \leq \sup_{t\in[-1,1]} \left|\sum_{j=0}^{n} a_j t^j - \sum_{j=0}^{n} b_j t^j\right| \leq \sum_{j=0}^{n} |a_j - b_j| \leq (n+1)\|\mathbf{a} - \mathbf{b}\|,$$

so F is continuous. Also

$$F(\mathbf{a}) \geq \sup_{t\in[-1,1]} \left|\sum_{j=0}^{n} a_j t^j\right| - \|f\|_\infty$$

so, by Corollary 8.3 (ii), we can find a $K > 0$ such that

$$\mathbf{a} \notin [-K, K]^{n+1} \Rightarrow F(\mathbf{a}) \geq F(\mathbf{0}).$$

By compactness, F attains a minimum at some point $\mathbf{p} \in [-K, K]^{n+1}$ and

$$P(t) = \sum_{j=0}^{n} p_j t^j$$

is the required polynomial. □

Proof of Lemma 9.1. As in Lemma 7.2, we take

$$e_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}.$$

If

$$\int_a^b P(x)\, dx = \sum_{j=0}^n A_j P(x_j)$$

for all polynomials of degree n or less, then, setting $P = e_k$, gives us

$$A_k = \int_a^b e_k(x)\, dx,$$

proving uniqueness.

On the other hand, if P has degree n or less,

$$Q = P - \sum_{j=0}^n P(x_j) e_j$$

has degree n or less but vanishes at the $n + 1$ points x_j. Thus $Q = 0$ and

$$P = \sum_{j=0}^n P(x_j) e_j,$$

whence

$$\int_a^b P(x)\, dx = \sum_{j=0}^n A_j P(x_j)$$

with

$$A_j = \int_a^b e_j(x)\, dx.$$

$\square$

Proof of Lemma 9.2. Linear independence shows that $\mathbf{v} \neq \mathbf{0}$. We have $\|\mathbf{e}_{n+1}\| = \|\mathbf{v}\|^{-1}\|\mathbf{v}\| = 1$. Now

$$\begin{aligned}\langle \mathbf{v}, \mathbf{e}_k \rangle &= \left\langle \mathbf{f} - \sum_{j=1}^n \langle \mathbf{f}, \mathbf{e}_j \rangle \mathbf{e}_j, \mathbf{e}_k \right\rangle \\ &= \langle \mathbf{f}, \mathbf{e}_k \rangle - \sum_{j=1}^n \langle \mathbf{f}, \mathbf{e}_j \rangle \langle \mathbf{e}_k, \mathbf{e}_j \rangle \\ &= \langle \mathbf{f}, \mathbf{e}_k \rangle - \langle \mathbf{f}, \mathbf{e}_k \rangle = 0\end{aligned}$$

so $\langle \mathbf{e}_{n+1}, \mathbf{e}_k \rangle = 0$ for all $1 \leq k \leq n$. $\square$

Proof of Lemma 9.4. Suppose that p_n has k roots α_j of *odd* order (that is to say the polynomial changes sign at the root) on $(-1,1)$. If we set $Q(t) = \prod_{j=1}^{k}(t-\alpha_j)$, then $p_n(t)Q(t)$ is a continuous single signed not everywhere zero function so

$$\int_{-1}^{1} Q(t)p_n(t)\,dt \neq 0.$$

Thus Q has degree at least n, so $k \geq n$.

It follows that $k = n$ and all of the roots of p_n are simple lying in $(-1,1)$. $\square$

Proof of Theorem 9.5. (i) By long division, $Q = p_nS + T$, where S and T are polynomials of degree at most $n-1$. Thus

$$\begin{aligned}\int_{-1}^{1} Q(x)\,dx &= \int_{-1}^{1} S(x)p_n(x)\,dx + \int_{-1}^{1} T(x)\,dx = \int_{-1}^{1} T(x)\,dx \\ &= \sum_{j=1}^{n} A_jT(\alpha_j) = \sum_{j=1}^{n} A_jT(\alpha_j) + \sum_{j=1}^{n} A_jp_n(\alpha_j)S(\alpha_j) = \sum_{j=1}^{n} A_jQ(\alpha_j).\end{aligned}$$

(ii) Let $P(x) = \prod_{j=1}^{n}(x-\beta_j)$. If R is a polynomial of degree $n-1$ or less, then RP has degree at most $2n-1$, so

$$\int_{-1}^{1} R(x)P(x)\,dx = \sum_{j=1}^{n} B_jR(\beta_j)P(\beta_j).$$

Thus $\langle P, R\rangle = 0$ for all polynomials of degree $n-1$ or less, so P is a scalar multiple of the nth Legendre polynomial p_n and the β_j are the roots p_n. $\square$

Proof of Theorem 9.6. (i) Let

$$P_k(x) = \left(\prod_{j\neq k} \frac{x-x_j}{x_k-x_j}\right)^2.$$

Then P_k has degree $2n-2$, so

$$0 < \int_{-1}^{1} P_k(x)\,dx = \sum_{j=1}^{n} A_jP_k(\alpha_j) = A_k.$$

(ii) Taking $P = 1$ in the formula, we obtain

$$2 = \int_{-1}^{1} 1\,dx = \sum_{j=1}^{n} A_j.$$

(iii) We have

$$\begin{aligned}
&\left|\int_{-1}^{1} f(x)\,dx - \sum_{j=1}^{n} A_j f(\alpha_j)\right| \\
&\qquad = \left|\int_{-1}^{1} \big(f(x) - P(x)\big)\,dx - \sum_{j=1}^{n} A_j\big(f(\alpha_j) - P(\alpha_j)\big)\right| \\
&\qquad \le \left|\int_{-1}^{1} \big(f(x) - P(x)\big)\,dx\right| + \left|\sum_{j=1}^{n} A_j\big(f(\alpha_j) - P(\alpha_j)\big)\right| \\
&\qquad \le \int_{-1}^{1} |f(x) - P(x)|\,dx + \sum_{j=1}^{n} A_j|f(\alpha_j) - P(\alpha_j)| \\
&\qquad \le 2\|f - P\|_\infty + \sum_{j=1}^{n} A_j\|f - P\|_\infty \le 4\|f - P\|_\infty.
\end{aligned}$$

(iv) Let $\epsilon > 0$. By Weierstrass's theorem, we can find a polynomial P such that $\|f - P\|_\infty \le \epsilon/4$. Then, if n is greater than the degree of P, part (iii) tells us that

$$\left|\int_{-1}^{1} f(x)\,dx - G_n f\right| \le 4\|f - P\|_\infty \le \epsilon.$$

□

Proof. (i) Let

$$P_k(x) = \left(\prod_{j\neq k} \frac{x - x_j}{x_k - x_j}\right)^2.$$

Then P_k has degree $2n - 2$, so

$$0 < \int_{-1}^{1} P_k(x)\,dx = \sum_{j=1}^{n} A_j P_k(\alpha_j) = A_k.$$

(ii) Taking $P = 1$ in the formula, we obtain

$$2 = \int_{-1}^{1} 1\,dx = \sum_{j=1}^{n} A_j.$$

(iii) We have

$$\begin{aligned}
&\left|\int_{-1}^{1} f(x)\,dx - \sum_{j=1}^{n} A_j f(\alpha_j)\right| \\
&\qquad = \left|\int_{-1}^{1} \big(f(x) - P(x)\big)\,dx - \sum_{j=1}^{n} A_j\big(f(\alpha_j) - P(\alpha_j)\big)\right| \\
&\qquad \leq \left|\int_{-1}^{1} \big(f(x) - P(x)\big)\,dx\right| + \left|\sum_{j=1}^{n} A_j\big(f(\alpha_j) - P(\alpha_j)\big)\right| \\
&\qquad \leq \int_{-1}^{1} |f(x) - P(x)|\,dx + \sum_{j=1}^{n} A_j|f(\alpha_j) - P(\alpha_j)| \\
&\qquad \leq 2\|f - P\|_\infty + \sum_{j=1}^{n} A_j\|f - P\|_\infty \leq 4\|f - P\|_\infty.
\end{aligned}$$

(iv) Let $\epsilon > 0$. By Weierstrass's theorem, we can find a polynomial P such that $\|f - P\|_\infty \leq \epsilon/4$. Then, if n is greater than the degree of P, part (iii) tells us that

$$\left|\int_{-1}^{1} f(x)\,dx - G_n f\right| \leq 4\|f - P\|_\infty \leq \epsilon.$$

□

Proof of Lemma 10.1. It is a standard observation about metric spaces (X, d) that, since $d(x, y) + d(y, z) \geq d(x, z)$, we have $d(y, z) \geq d(x, z) - d(x, y)$ and similarly $d(y, z) = d(z, y) \geq d(x, y) - d(x, z)$, so that

$$d(y, z) \geq |d(x, z) - d(x, y)|.$$

Thus, if we write $f(\mathbf{x}) = \|\mathbf{a} - \mathbf{x}\|$, we have

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \|\mathbf{x} - \mathbf{y}\|,$$

so f is continuous and attains its minimum on the compact set E. □

Solution of Exercise 10.2. (i) Consider $n = 2$,

$$E = \{(x, y) \,:\, x^2 + y^2 = 1\} \text{ and } \mathbf{e} = \mathbf{0}.$$

Any point of E will do.

(ii) Suppose that E is convex, $\mathbf{e},\ \mathbf{f} \in E$ and $\|\mathbf{a}-\mathbf{e}\| = \|\mathbf{a}-\mathbf{f}\|$. Then

$$\frac{\mathbf{e}+\mathbf{f}}{2} \in E,$$

but the parallelogram law tells us that

$$\begin{aligned} 4\|\mathbf{a}-\mathbf{e}\|^2 &= 2\|\mathbf{a}-\mathbf{e}\|^2 + \|\mathbf{a}-\mathbf{f}\|^2 \\ &= \|(\mathbf{a}-\mathbf{e})+(\mathbf{a}-\mathbf{f})\|^2 + \|(\mathbf{a}-\mathbf{e})-(\mathbf{a}-\mathbf{f})\|^2 \\ &= 4\left\|\mathbf{a}-\frac{\mathbf{e}+\mathbf{f}}{2}\right\|^2 + \|\mathbf{e}-\mathbf{f}\|^2 \end{aligned}$$

and so

$$\left\|\mathbf{a}-\frac{\mathbf{e}+\mathbf{f}}{2}\right\| \leq \|\mathbf{a}-\mathbf{e}\|$$

with equality only if $\mathbf{e}=\mathbf{f}$. (Alternatively draw a diagram and use a little school geometry to obtain the same result.) $\square$

Proof of Lemma 10.3. (i) Recall that, if $\mathbf{u} \in \mathbb{R}^n$, then we can find $\mathbf{v} \in F$ such that $\|\mathbf{u}-\mathbf{v}\| = d(\mathbf{u}, F)$. If $\mathbf{u}' \in \mathbb{R}^n$, then

$$d(\mathbf{u}', F) \leq \|\mathbf{u}'-\mathbf{v}\| \leq \|\mathbf{u}'-\mathbf{u}\| + \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}'-\mathbf{u}\| + d(\mathbf{u}, F).$$

The same argument shows that $d(\mathbf{u}, F) \leq \|\mathbf{u}'-\mathbf{u}\| + d(\mathbf{u}', F)$. Thus

$$|d(\mathbf{u}, F) - d(\mathbf{u}', F)| \leq \|\mathbf{u}'-\mathbf{u}\|.$$

and the map $\mathbf{u} \mapsto d(\mathbf{u}, F)$ is continuous. By compactness, it attains its minimum on E and this is the required result.

(ii) Choose $\mathbf{u} \in E$. Since F is bounded we can find an R such that $B(\boldsymbol{u}, R) \supseteq F$. Let

$$E^* = \bar{B}(\mathbf{u}, 2R+1) \cap E.$$

If $\mathbf{e} \in E \setminus E^*$, then $d(\mathbf{e}, F) \geq d(\mathbf{u}, F) + 1$.

Since E^* is compact, part (i) tells us that there exist a $\mathbf{e} \in E^*$ and a $\mathbf{f} \in F$ such that

$$\|\mathbf{e}-\mathbf{f}\| = \inf_{\mathbf{y}\in E^*} d(\mathbf{y}, F)$$

and so by the previous paragraph

$$\|\mathbf{e}-\mathbf{f}\| = \inf_{\mathbf{y}\in E} d(\mathbf{y}, F)$$

(iii) Let $n=1$, $E = \{r+1/r \,:\, r \in \mathbb{Z},\ r \geq 2\}$ and $F = \{r \,:\, r \in \mathbb{Z},\ r \geq 2\}$. We have E and F closed and $\tau(E, F) = 0$, but $|e-f| > 0$ for all $e \in E$, $f \in F$. $\square$

Solution of Exercise 10.4. Repeat the counter-example of Exercise 10.2. Take $n = 2$,
$$E = \{(x, y) \,:\, x^2 + y^2 = 1\}, \; F = \{\mathbf{0}\}.$$
$\square$

Solution of Exercise 10.5. (i) follows directly from the definition. (Alternatively take the $\mathbf{e}$ and $\mathbf{f}$ of Lemma 10.3 (i) and observe that $\tau(E, f) = \|\mathbf{e} - \mathbf{f}\| \geq 0$.)

Lemma 10.3 (i) also shows that
$$\tau(F, E) \leq \|\mathbf{e} - \mathbf{f}\| = \tau(E, F).$$
Interchanging E and F, yields $\tau(E, F) \leq \tau(F, E)$ so $\tau(E, F) = \tau(F, E)$.

If we work with $n = 1$, setting $E = \{0\}$, $F = \{0, 1\}$, gives $\tau(E, F) = 0$, but $E \neq F$.

If we work with $n = 1$, then setting $E = \{0\}$, $F = \{0, 1\}$, $G = \{1\}$ gives $\tau(E, F) + \tau(F, G) = 0 + 0 = 0$, but $\tau(E, G) = 1$. $\square$

Solution of Exercise 10.6. In our proof of Lemma 10.3 (i) we showed that the map $\mathbf{u} \mapsto d(\mathbf{u}, F)$ is continuous. It follows that it attains its maximum on the compact set E. $\square$

Solution of Exercise 10.7. Since $d(\mathbf{e}, F) \geq 0$ for all $\mathbf{e}$, we have $\sigma(E, F) \geq 0$.

If $n = 1$, $E = \{0\}$, $F = [0, 1]$, then $\sigma(E, F) = 0$, but $\sigma(F, E) = 1$, so conditions (ii) and (iii) fail.

$$\sigma(E, F) = 0 \Leftrightarrow d(\mathbf{e}, F) = 0 \;\forall \mathbf{e} \in E \Leftrightarrow \mathbf{e} \in F \;\forall \mathbf{e} \in E \Leftrightarrow E \subseteq F.$$
$\square$

Proof of Lemma 10.8. Given $\mathbf{e} \in E$, we can find $\mathbf{f} \in F$ such that $\|\mathbf{e} - \mathbf{f}\| = d(\mathbf{e}, F)$. If $\mathbf{g} \in G$, then
$$\begin{aligned} d(\mathbf{e}, G) &\leq \|\mathbf{e} - \mathbf{g}\| \leq \|\mathbf{e} - \mathbf{f}\| + \|\mathbf{f} - \mathbf{g}\| \\ &= d(\mathbf{e}, F) + \|\mathbf{f} - \mathbf{g}.\| \end{aligned}$$
Since $\mathbf{g} \in G$ was arbitrary,
$$d(\mathbf{e}, G) \leq d(\mathbf{e}, F) + d(\mathbf{f}, G) \leq \sigma(E, F) + \sigma(F, G)$$
and so
$$\sigma(E, G) \leq \sigma(E, F) + \sigma(F, G).$$
$\square$

Proof of Theorem 10.10. Observe that

$$\begin{aligned}
\rho(E,F) &= \sigma(E,F)+\sigma(F,E) \geq 0 \\
\rho(E,F) = 0 \Leftrightarrow \sigma(E,F) &= \sigma(F,E) = 0 \Leftrightarrow E \subseteq F,\ F \subseteq E \Leftrightarrow E = F \\
\rho(E,F) &= \sigma(E,F)+\sigma(F,E) = \sigma(F,E)+\sigma(E,F) = \rho(F,E) \\
\rho(E,F)+\rho(F,G) &= \sigma(E,F)+\sigma(F,G)+\sigma(G,F)+\sigma(F,E) \\
&\geq \sigma(E,G)+\sigma(G,E) = \rho(E,G),
\end{aligned}$$

as desired. □

Proof of Theorem 10.12. (i) This part may be familiar from 1B. (Indeed the reader may well be able to supply a more sophisticated proof.) Since the intersection of closed sets is closed and the intersection of bounded sets is bounded we only have to show that K is non-empty.

Choose $\mathbf{x}_n \in K_n$. Since K_1 is compact and $\mathbf{x}_n \in K_1$ for every n we can find an $\mathbf{x} \in K_1$ and $n(j) \geq j$ such that $\mathbf{x}_{n(j)} \to \mathbf{x}$ (in the Euclidean metric) as $j \to \infty$

Automatically,

$$\mathbf{x}_{n(j)} \in K_{n(j)} \subseteq K_j \subseteq K_p$$

for all $j \geq p$, so, since K_p is closed, $\mathbf{x} \in K_p$ for all $p \geq 1$. It follows that $\mathbf{x} \in K$ and K is non-empty.

(ii) Since $K \subseteq K_p$ it follows that

$$\rho(K, K_p) = \sup_{\mathbf{e}\in K_p} \inf_{\mathbf{k}\in K} \|\mathbf{e}-\mathbf{k}\|$$

and, in particular that $\rho(K,K_p)$ is a decreasing positive sequence.

Thus if $K_p \underset{\rho}{\not\to} K$ there must exist an $\eta > 0$ with

$$\rho(K, K_p) \geq 2\eta$$

and there must exist $\mathbf{k}_p \in K_p$ with

$$\|\mathbf{k}_p - \mathbf{k}\| \geq \eta$$

for all $\mathbf{k} \in K$.

Since K_1 is compact and $\mathbf{k}_p \in K_1$ for every p we can find an $\mathbf{x} \in K_1$ and $p(j) \geq j$ such that $\mathbf{k}_{p(j)} \to \mathbf{x}$ (in the Euclidean metric) as $j \to \infty$. As in part (i), we know that $\mathbf{x} \in K$ so

$$\|\mathbf{k}_{p(j)} - \mathbf{x}\| \geq \eta.$$

for all j giving us a contradiction.

Part (ii) follows by reductio ad absurdum. □

Proof of Lemma 10.13. If $\mathbf{z}_n \in K + \bar{B}(0,r)$, then $\mathbf{z}_n = \mathbf{x}_n + \mathbf{y}_n$ with $\mathbf{x}_n \in K$, $\|\mathbf{y}_n\| \leq r$. By compactness, we can first extract a convergent subsequence $\mathbf{x}_{n(j)} \in K$ and then a convergent subsequence $\mathbf{y}_{n(j(k))} \in \bar{B}(0,r)$. It follows that $\mathbf{z}_{n(j(k))} = \mathbf{x}_{n(j(k))} + \mathbf{y}_{n(j(k))}$ converges to a point in $K + \bar{B}(0,r)$ so we are done. □

Proof of Theorem 10.11. By Lemma 1.11, it suffices to show that, if we have sequence of non-empty compact sets with $\rho(E_n, E_{n+1}) < 8^{-n}$ for $n \geq 1$, then the sequence converges. Set

$$K_n = E_n + \bar{B}(0, 6 \times 8^{-n}).$$

Then K_n is compact and $\rho(E_n, K_n) = 6 \times 8^{-n}$, so it is sufficient to show that K_n converges.

To do this, we observe that $K_{n+1} \subseteq K_n$ and so we may apply Theorem 10.12. □

Proof for Example 11.1. Observe that, taking C to be the contour $z = e^{i\theta}$ as θ runs from 0 to 2π, we have

$$\begin{aligned}\sup_{z \in \bar{D}} |f(z) - p(z)| &\geq \frac{1}{2\pi} \left| \int_C f(z) - p(z)\, dz \right| \\ &= \frac{1}{2\pi} \left| \int_C f(z)\, dz \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} e^{-i\theta} i e^{i\theta}\, d\theta \right| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} i\, d\theta \right| = 1.\end{aligned}$$

□

Proof for Example 11.3. By exactly the same computations as in Example 11.1,

$$\sup_{z \in T} |f(z) - p(z)| \geq \frac{1}{2\pi} \left| \int_C f(z) - p(z)\, dz \right| = 1.$$

□

Solution for Exercise 11.9. (i) Just observe that

$$\frac{1}{z} = \frac{1}{w + (z - w)} = \frac{1}{w(1 + (z-w)/w)} = \sum_{j=0}^{\infty} \frac{(-1)^j (z-w)^j}{w^{j+1}}$$

for $|(z-w)/w| < 1$.

(ii) It is easy to check that Ω is open and bounded. To see that Ω is connected, suppose that $w_1, w_2 \in \Omega$. Then we can find r_k and θ_k with

$10^{-2} < r_k < 1$ and $-\pi < \theta < \pi$ such that $w_k = r_k e^{i\theta_k}$ $[k = 1,\ 2]$. If we define $\gamma : [0, 1] \to \Omega$ by

$$\gamma(t) = \big((1-t)r_1 + tr_2\big) \exp\big(i(1-t)\theta_1 + it\theta_2\big),$$

then γ gives a path from w_1 to w_2.

Suppose, if possible, that

$$z^{-1} = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

for all $z \in \Omega$. Then the power series converges on some open disc D centre z_0 with $D \supseteq \Omega$. Thus $D \supseteq \{z : |z| < 1\}$. By Lemma 11.8,

$$z^{-1} = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

for all z with $0 < |z| < 1$. Allowing $z \to 0$, gives a contradiction. □

Proof of Lemma 11.12. We may suppose K non-empty. Since K is compact, $\mathbb{C} \backslash \Omega$ closed and the two sets are disjoint, it follows that $\eta = \tau(K, \mathbb{C} \backslash \Omega)/8 > 0$ (i.e. $|k - w| > 8\eta$ for all $k \in K$, $w \notin \Omega$).

Consider a grid of squares side η. We consider the collection Γ of closed squares S lying entirely within Ω with boundary contours $C(S)$. Observe that

$$\bigcup_{S \in \Gamma} C(S) \supseteq \{k + u : k \in K,\ |u| \leq 2\eta\}$$

By Cauchy's theorem

$$f(z) = \frac{1}{2\pi i} \sum_{S \in \Gamma} \int_{C(S)} \frac{f(w)}{w - z}\, dw$$

for all $z \in K$ such that z does not lie on the boundary of some S. By cancelling internal sides,

$$f(z) = \sum_{m=1}^{M} \int_{C_m} \frac{f(w)}{w - z}\, dw \qquad \bigstar$$

with the piece-wise linear contours C_m $[1 \leq m \leq M]$ lying entirely within $\Omega \setminus K$.

We deal with the case when z lies on the boundary of some S by erasing any sides through z and repeating the argument with the new (non-regular) grid. (Alternatively, we could observe that both sides of equation $\bigstar$ are continuous.) □

Proof of Lemma 11.13. Observe that K and $\bigcup_{m=1}^{M} C_m$ are compact and disjoint. $\square$

Proof of Lemma 11.14. By Lemma 11.13 it is sufficient to show that, if C is a straight line segment joining lying in $\Omega \setminus K$, then, given ϵ, we can find $B_m \in \mathbb{C}$ and $\beta_m \in C$ with

$$\left| \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw - \sum_{m=1}^{M} \frac{B_m}{z - \beta_m} \right| < \epsilon$$

for all $z \in K$.

To this end, note that

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw = \int_0^1 F(t, z)\, dt$$

where $F : [0, 1] \times K \to \mathbb{C}$ is defined by

$$F(t, z) = \frac{1}{2\pi i} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)$$

with $\gamma(t) = (1 - t)z_1 + tz_2$. Since $[0, 1] \times K$ is compact and F is continuous, F must be uniformly continuous so there exists a $\delta > 0$ such that

$$|F(t, z) - F(s, z)| < \epsilon \text{ for all } |t - s| < \delta.$$

If we choose an integer $M > \delta^{-1}$ and set $F_M(t, z) = F(m/M, z)$ whenever $(m - 1)/M < t \le m/M$ $[1 \le m \le M]$, then $|F(t, z) - F_M(t, z)| \le \epsilon$ so

$$\left| \int_0^1 F(t, z)\, dt - \int_0^1 F_M(t, z)\, dt \right| \le \epsilon.$$

Since

$$\int_0^1 F_M(t, z)\, dt = \sum_{m=1}^{M} \frac{B_m}{z - \beta_m}$$

for appropriate B_m and β_m, we are done. $\square$

Proof of Theorem 11.11 from Lemma 11.15. We use the result and notation of Lemma 11.14. Choose polynomials P_n such that

$$\left| P_n(z) - \frac{1}{z - \alpha_n} \right| \le \frac{\epsilon}{(N + 1)(|A_n| + 1)}$$

for all $z \in K$. Then, if

$$P(z) = \sum_{n=1}^{N} A_n P_n(z),$$

P is a polynomial and

$$\begin{aligned}|f(z) - P(z)| &\leq \left| f(z) - \sum_{n=1}^{N} \frac{A_n}{z - \alpha_n} \right| + \sum_{n=1}^{N} |A_n| \left| P_n(z) - \frac{1}{z - \alpha_n} \right| \\ &\leq \epsilon + N \frac{\epsilon}{N+1} \leq 2\epsilon.\end{aligned}$$

Since ϵ was arbitrary the result follows. □

Proof of Lemma 11.17. Since K is compact, it is bounded and we can find an $R > 0$ such that $|z| < R/2$ whenever $z \in K$. The standard geometric series result shows that if $|\alpha| > R$

$$\frac{-1}{\alpha} \sum_{r=0}^{n} \frac{z^r}{\alpha^r} \to \frac{-1}{\alpha} \times \frac{1}{1 - (z/\alpha)} = \frac{1}{z - \alpha}$$

uniformly for $|z| \leq R/2$ and so for $z \in K$. □

Proof of Lemma 11.18. Since $\alpha \in \Lambda(K)$ we know that there exists a sequence of polynomials P_n such that

$$P_n(z) \to \frac{1}{z - \alpha}$$

uniformly on K. Moreover, since (by compactness) $z \mapsto (z-\alpha)^{-1}$ is bounded on K, the P_n are uniformly bounded.

On the other hand,

$$\frac{1}{z - \beta} = \frac{1}{z - \alpha - (\beta - \alpha)} = \frac{1}{z - \alpha} \times \frac{1}{1 - \frac{\beta - \alpha}{z - \alpha}}.$$

Since

$$\left| \frac{\beta - \alpha}{z - \alpha} \right| \leq \frac{|\beta - \alpha|}{d(\alpha, K)} < 1$$

for all $z \in K$, we know that, given $\epsilon > 0$, there exists an N with

$$\left| \frac{1}{z - \beta} - \sum_{j=0}^{N} \frac{(\beta - \alpha)^j}{(z - \alpha)^{j+1}} \right| < \epsilon/2$$

for all $z \in K$. By the first paragraph, we can find an M such that

$$\left| \frac{(\beta-\alpha)^j}{(z-\alpha)^{j+1}} - (\beta-\alpha)^j P_M(z)^j \right| < \epsilon/(2N+4)$$

for each $0 \leq j \leq N$ and so

$$\left| \frac{1}{z-\beta} - \sum_{j=0}^{N} (\beta-\alpha)^j P_M(z)^j \right| < \epsilon$$

for all $z \in K$. We have shown that $\beta \in \Lambda(K)$. $\square$

Proof of Lemma 11.19. Let $a \in \mathbb{C} \setminus K$. By Lemma 11.17, $\Lambda(K)$ is non-empty so we may choose a $b \in \Lambda(K)$. Since $\mathbb{C} \setminus K$ is path connected we can find a continuous $\gamma : [0,1] \to \mathbb{C} \setminus K$ with $\gamma(0) = b$, $\gamma(1) = a$. The continuous image of a compact set is compact and $\gamma([0,1]) \cap K = \varnothing$ so (see Lemma 10.3) there exists a $\delta > 0$ such that $|\gamma(t) - k| > \delta$ for all $k \in K$ and all $t \in [0,1]$.

By uniform continuity, we can find an N such that

$$|s-t| \leq 1/N \Rightarrow |\gamma(t) - \gamma(s)| < \delta/2.$$

Writing $x_r = \gamma(r/N)$, we see that $x_0 = b \in \Lambda(K)$ and, applying Lemma 11.18,

$$x_{r-1} \in \Lambda(K) \Rightarrow x_r \in \Lambda(K)$$

for $1 \leq r \leq N$. Thus $a = x_N \in \Lambda(K)$ and we are done. $\square$

Proof of Example 11.20. Consider the map T_n given by

$$T_n(z) = (2^{-n} + z)\exp(-2^{-n} i\pi)$$

(a translation followed by a rotation).

Let $g_n = T_n^{-1} f T_n$ and

$$l_n = \{r \exp(i2^{-n}) - 2^{-n} \,:\, r \in \mathbb{R},\ r \geq 0\} = T_n^{-1}\{x \,:\, x \in \mathbb{R},\ x \geq 0\}$$

so g_n is analytic on $\mathbb{C} \setminus l_n$. We see that $g_n(z) \to f(z)$ pointwise as $n \to \infty$. (The reader may find it convenient to examine the case when $z = x$ with x real and $x \geq 0$ separately.)

We now set

$$U_n = l_n + \operatorname{Int} D(0, 2^{-8n}) = \{z + w, :\ z \in l_n,\ |w| < 2^{-8n}\}$$

so that U_n is open and $U_n \cap U_m = \varnothing$ for $n \neq m$. Finally we take

$$K_n = \operatorname{Cl} D \setminus U_n$$

so that K_n is compact. By Runge's theorem we can find a polynomial P_n such that $|P_n(z) - g_n(z)| \leq 2^{-n}$ for all $z \in K_n$.

Now choose a particular $z \in D$. We know that there exists an N (depending on z) such that $z \in K_n$ for all $n \geq N$. Considering only $n \geq N$, we have $|P_n(z) - g_n(z)| \to 0$ and (as we observed in the first paragraph) $g_n(z) \to f(z)$ so $P_n(z) \to f(z)$ as $n \to \infty$. Since z was arbitrary, we are done. $\square$

Proof of Theorem 12.1. This is probably familiar from 1A.

Suppose, if possible, that $e = p/q$ with p and q integers and $q \geq 2$. Then

$$q!e \text{ and } q!\sum_{r=0}^{q}\frac{1}{r!}$$

are integers so

$$M = q!\sum_{r=q+1}^{\infty}\frac{1}{r!} = q!e - q!\sum_{r=0}^{q}\frac{1}{r!}$$

is an integer. But

$$0 < M = q!\sum_{r=q+1}^{\infty}\frac{1}{r!} \leq \sum_{r=q+1}^{\infty}\frac{1}{q^{r-q}} = \frac{1}{q}\times\frac{1}{1-q^{-1}} = \frac{1}{q-1} < 1$$

and there is no integer strictly between 0 and 1. Our assumption has led to a contradiction so e must be irrational. $\square$

Proof of Lemma 12.3. Observe that

$$f_n(x) = \sum_{s=0}^{n}\binom{n}{s}\pi^{n-s}x^{n+s}$$

Thus

$$f^{(r)}(0) = 0$$

if $0 \leq r \leq n-1$ or $2n+1 \leq r$ and

$$f^{(n+r)}(0) = (n+r)!\binom{n}{r}\pi^{n-r}$$

for $0 \leq r \leq n$. By symmetry about $\pi/2$,

$$f^{(r)}(\pi) = (-1)^r f^{(r)}(0).$$

Thus $f^{(r)}(0)$ and $f^{(r)}(\pi)$ always take the form of $M \times n! \times \pi^k$ where M is an integer and k is an integer with $0 \leq k \leq n$.

Now integration by parts gives

$$\begin{aligned}\int_0^\pi f^{(m)}(x)\cos x\,dx &= \left[f^{(m)}(x)\sin x\right]_0^\pi - \int_0^\pi f^{(m+1)}(x)\sin x\,dx \\ &= -\int_0^\pi f^{(m+1)}(x)\sin x\,dx\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi f^{(m)}(x)\sin x\,dx &= -\left[f^{(m)}(x)\cos x\right]_0^\pi + \int_0^\pi f^{(m+1)}(x)\cos x\,dx \\ &= (f^{(m)}(\pi) - f^{(m)}(0)) + \int_0^\pi f^{(m+1)}(x)\cos x\,dx.\end{aligned}$$

Thus integration by parts $2n+1$ times gives

$$\int_0^\pi f(x)\sin x\,dx = n!U(\pi)$$

where U is a polynomial of degree at most n with integer coefficients. □

Proof of Theorem 12.2 from Lemma 12.3. Suppose that $\pi = p/q$ with p and q integers and $q \geq 1$. It follows from Lemma 12.3 that

$$\frac{q^n}{n!}\int_0^\pi f_n(x)\sin x\,dx = q^n\sum_{j=0}^n a_j\pi^j = \sum_{j=0}^n a_j q^{n-j}p^j \in \mathbb{Z}.$$

But (by school calculus or completing the square or the AM-GM inequality) $x(\pi - x)$ takes its maximum when $x = \pi/2$ so

$$0 \leq f_n(x) \leq (\pi/2)^{2n}$$

and, since $f_n(x)\sin x$ is strictly positive for $0 < x < \pi$,

$$0 < \int_0^\pi f_n(x)\sin x\,dx \leq \int_0^\pi (\pi/2)^{2n}\,dx = \pi^{2n+1}2^{-2n}.$$

Thus

$$0 < \frac{q^n}{n!}\int_0^\pi f_n(x)\sin x\,dx \leq \frac{1}{n!}\pi^{2n+1}2^{-2n}q^n < 1$$

for n sufficiently large.

However there is no integer strictly between 0 and 1. Our assumption has led to a contradiction. Thus π is irrational. □

Solution of Exercise 12.5. Only if is trivial since integers are rational numbers.

To see *if*, observe that, if α satisfies

$$\sum_{j=0}^{N} \frac{p_j}{q_j} \alpha^j = 0$$

with p_j, q_j integers, $q_j \neq 0$ for all j, $p_N \neq 0$, $N \geq 1$, then

$$\sum_{j=0}^{N} p_j \prod_{i \neq j} q_i \alpha^j = 0.$$

□

Proof of Lemma 12.6. This was done in 1A. There are only finitely many polynomials of the form

$$\sum_{j=0}^{N} a_j x^j$$

with $n \geq N \geq 1$, $a_N \neq 0$ and all a_j integers with $|a_j| \leq n$. A polynomial has only finitely many roots, so the set E_n of roots of such polynomials is finite so countable. Thus $E = \bigcup_{n=1}^{\infty} E_n$ is the countable union of countable sets so countable. But E is the set of algebraic numbers, so we are done. □

Proof of Theorem 12.7. Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

Since a polynomial has only finitely many roots, we can find an $R \geq 1$ such that all the roots of P lie in $[-R+1, R-1]$. If we take $0 < c \leq 1$, the required result will be automatic for $p/q \notin [-R, R]$.

Now P' is continuous, so, by compactness, there exists an $M > 1$ such that $|P'(t)| \leq M$ for $t \in [-R, R]$. (We could also prove this directly.) If α is an irrational root, p, $q \in \mathbb{Z}$ with $q \neq 0$, $p/q \in [-R, R]$ and $P(p/q) \neq 0$, then the mean value theorem yields

$$|P(\alpha) - P(p/q)| \leq M \left| \alpha - \frac{p}{q} \right|$$

so, since $P(\alpha) = 0$,

$$|P(p/q)| \leq M \left| \alpha - \frac{p}{q} \right|.$$

Now $q^n P(p/q)$ is a non-zero integer, so $|q^n P(p/q)| \geq 1$ and

$$q^{-n} \leq M \left| \alpha - \frac{p}{q} \right|,$$

that is to say

$$M^{-1} q^{-n} \leq \left| \alpha - \frac{p}{q} \right|.$$

Since there are only a finite number of roots and so only a finite number of irrational roots, we know that there is a $c' > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| \geq c' q^{-n}$$

whenever α is an irrational root, p, $q \in \mathbb{Z}$ with $q \neq 0$ and $P(p/q) = 0$.

Taking $c = \min\{M^{-1}, c', 1\}$, we have the required result. □

Proof of Theorem 12.8. Let

$$L = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}.$$

We observe that L is irrational since its decimal expansion is not recurring. If $q_m = 10^{m!}$ and

$$p_m = q_m \sum_{n=0}^{m} \frac{1}{10^{n!}},$$

then p_m and q_m are integers with $q_m \neq 0$.

We observe that

$$\left| L - \frac{p_m}{q_m} \right| = \sum_{j=m+1}^{\infty} \frac{1}{10^{j!}} \leq \frac{1}{10^{(m+1)!}} \sum_{j=0}^{\infty} \frac{1}{10^j} \leq \frac{2}{10^{(m+1)!}}$$

and, given any $c > 0$ and any integer $n \geq 1$, we can find an m such that

$$\left| L - \frac{p_m}{q_m} \right| \leq \frac{2}{10^{(m+1)!}} < \frac{c}{q^n}.$$

Thus Theorem 12.7 tells us that L is transcendental. □

Solution of Exercise 12.9. Essentially the same argument as for Theorem 12.8 tells us that

$$\sum_{n=0}^{\infty} \frac{b_j}{10^{n!}}$$

with $b_j \in \{1,\, 2\}$ is transcendental.

The map
$$\theta : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$$
given by
$$\theta(\boldsymbol{\zeta}) = \sum_{n=0}^{\infty} \frac{\zeta(j)+1}{10^{n!}}$$
is injective and its image (as we have just seen) consists of transcendental numbers. Since, as we saw in 1A, the set $\{0,1\}^{\mathbb{N}}$ is uncountable and θ is injective, $\theta(\mathbb{N})$ is uncountable and we are done. □

Proof of Theorem13.1. We construct $x_j \in X$ and $\delta_j > 0$ inductively as follows. Choose any $x_0 \in X$ and set $\delta_0 = 1$.

Suppose that x_j and δ_j have been found. Since $X \setminus U_{j+1}$ has empty interior, we can find an $x_{j+1} \in U_{j+1}$ with $d(x_{j+1}, x_j) \leq \delta_j/4$. Since U_{j+1} is open we can find a $\delta_{j+1} > 0$ with $\delta_{j+1} \leq \delta_j/4$ such that $B(x_{j+1}, \delta_{j+1}) \subseteq U_{j+1}$.

By induction, $\delta_{j+k} \leq 4^{-k}\delta_j$ for $j, k \geq 0$, so, if $m \geq n \geq 0$,
$$\begin{aligned} d(x_n, x_m) &\leq \sum_{r=0}^{m-n-1} d(x_{n+r}, x_{n+r+1}) \\ &\leq \sum_{r=0}^{m-n-1} \delta_{n+r}/4 \leq \sum_{r=0}^{m-n-1} \delta_n 4^{-r-1} \leq \delta_n/2, \end{aligned}$$
so the sequence x_n is Cauchy and so converges to some point a.

We observe that
$$d(x_n, a) \leq d(x_n, x_m) + d(x_m, a) \leq \delta_n/2 + d(x_m, a) \to \delta_n/2$$
as $m \to \infty$. Thus
$$a \in B(x_j, \delta_j) \subseteq U_j$$
for all $j \geq 1$ and
$$a \in \bigcap_{j=1}^{\infty} U_j.$$
The result is proved □

Proof of the equivalence of Theorems 13.1 and 13.2. Set $F_j = X \setminus U_j$. □

Proof of the equivalence of Theorems 13.1 and 13.3. Let x have the property P_j if and only if $x \notin U_j$. □

Proof of Lemma 13.5. (i) This is just a restatement of Theorem 13.2.

(ii) The countable union of countable sets is countable. □

Proof of Theorem 13.7. Suppose that (E, d) is a non-empty countable complete space with no isolated points. Then each $\{e\}$ with $e \in E$ is closed (since singletons are always closed in metric spaces). However, since e not isolated, $B(x, \delta) \not\subseteq \{e\}$ for all $\delta > 0$, so $\{e\}$ is not open and $\{e\}$ has empty interior. Thus E is the countable union of closed sets $\{e\}$ with empty interior contradicting Theorem 13.2. The required result follows by reductio ad absurdum. □

Proof of Corollary 13.8. Observe that $\mathbb{R}$ with the usual metric is complete without isolated points. Theorem 13.7 now tells us that $\mathbb{R}$ is uncountable. □

Proof of Theorem 13.9. Banach's clever idea is to consider the set E_m consisting of all those $f \in C([0, 1])$ such that there exists an $x \in [0, 1]$ with the property

$$|f(x) - f(y)| \leq m|x - y|$$

for all $y \in [0, 1]$. Our proof falls into several parts.

(a) We show that, if f is differentiable at some point $x \in [0, 1]$, then there exists a positive integer m such that $f \in E_m$. It will then follow that any $g \in C([0, 1]) \setminus \bigcup_{m=1}^{\infty} E_m$ is nowhere differentiable.

To this end, suppose that f is differentiable at x. We can find an $\epsilon > 0$ such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq 1$$

for all $0 < |h| < \epsilon$, when $x + h \in [0, 1]$. Thus

$$|f(x+h) - f(x)| \leq (|f'(x)| + 1)|h|$$

for all $0 < |h| < \epsilon$ when $x + h \in [0, 1]$. We thus have

$$|f(x) - f(y)| \leq (|f'(x)| + 1)|x - y|$$

for all $y \in [0, 1]$ such that $|y - x| < \epsilon$. If we choose m with $m \geq |f'(x)| + 1$ and $m \geq 2K\epsilon^{-1}$, we will have $f \in E_m$.

(b) We now show that E_m is closed.

Suppose that $f_n \in E_m$ and $\|f_n - f\|_\infty \to 0$. By definition, there exists an $x_n \in [0, 1]$ with the property

$$|f_n(x_n) - f(y)| \leq m|x_n - y|$$

for all $y \in [0,1]$. By the Bolzano–Weierstrass property, we can find $x \in [0,1]$ and $n(r) \to \infty$ such that $x_{n(r)} \to x$ as $r \to \infty$.

Let $y \in [0,1]$. We have

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f(x_{n(r)})| + |f(x_{n(r)}) - f_{n(r)}(x_{n(r)})| \\ &\qquad + |f_{n(r)}(x_{n(r)}) - f_{n(r)}(y)| + |f_{n(r)}(y) - f(y)| \\ &\leq 2\|f - f_{n(r)}\|_\infty + |f(x) - f(x_{n(r)})| + m|x_{n(r)} - y| \\ &\to 0 + 0 + m|x - y| = m|x - y|.\end{aligned}$$

Since y was arbitrary, $f \in E_m$.

(c) Next we show that E_m has a dense complement.

Suppose that $f \in C([0,1])$ and $\epsilon > 0$. By Weierstrass's theorem on polynomial approximation (see Theorem 7.9), we can find a polynomial P such that

$$\|f - P\|_\infty \leq \epsilon/3.$$

Since P is continuously differentiable, there is a K such that $|P'(t)| \leq K$ for all $t \in [0,1]$. By the mean value theorem, it follows that

$$|P(x) - P(y)| \leq K|x - y|$$

for all x, $y \in [0,1]$.

Let $g(t) = P(t) + (\epsilon/3)\cos 2\pi N t$. Automatically,

$$\|g - f\|_\infty \leq \|f - P\|_\infty + \epsilon/3 \leq 2\epsilon/3 < \epsilon.$$

We claim that, provided only that N is large enough, $g \notin E_m$.

To see this choose r an integer with $0 \leq r \leq N - 1$ such that $0 \leq x - r/N \leq 1/N$. We have

$$\begin{aligned}&\max\{|g(r/N) - g(x)|, |g((r+1)/N) - g(x)|\} \\ &\qquad\geq \frac{|g(r/N) - g(x)| + |g((r+1)/N) - g(x)|}{2} \\ &\qquad\geq \frac{|g(r/N) - g((r+1)/N)|}{2} \\ &\qquad\geq \frac{2\epsilon/3 - |P(r/N) - P((r+1)/N)|}{2} \\ &\qquad\geq \epsilon/3 - K/N \geq \epsilon/6 \geq 4m/N.\end{aligned}$$

Thus at least one of the statements

$$|g(r/N) - g(x)| > m|r/n - x| \text{ or } |g((r+1)/N) - g(x)| > m|(r+1)/n - x|$$

is true for N sufficiently large (with N not depending on the choice of x).

(d) Thus $\bigcup_{m=1}^{\infty} E_m$ is a set of first category and we are done. □

Proof of Corollary 13.10. Observe that a closed subset of a complete metric space is complete under the inherited metric and that $\mathbb{R}$ is complete under the standard metric. $\square$

Proof of Lemma 13.11. (i) Suppose that $E_n \in \mathcal{E}_k$ and $E_n \to E$ in the Hausdorff metric. By definition, we can find $x_n \in E_n$ with $B(x_n, 1/k) \cap E = \{x_n\}$. By Bolzano–Weierstrass, we can find $n(j) \to \infty$ and $x \in [0,1]$ such that $|x_n(j) - x| \to 0$. We observe that $x \in E$.

Suppose, if possible, that $B(x, 1/k) \cap E \neq \{x\}$. Then we can find a $y \in E$ such that $|x - y| < 1/k$. Set $\delta = (1/k - |x - y|)/2$. Since $E_n \to E$ and $n(j) \to \infty$, we can find a J such that the Hausdorff distance $\rho(E_{n(J)}, E) < \delta$ and so there exists a $y' \in E_{n(J)}$ with $|y' - y| < \delta$ and so with $|x_{n(J)} - y'| < 1/k$, contrary to our hypothesis.

Thus $E \in \mathcal{E}_k$ and $\mathcal{E}_k$ is closed.

(ii) Let $G \in \mathcal{K}$ and let $\epsilon > 0$. Choose an integer $N > 5(\epsilon^{-1} + k + 1)$. Let

$$F = \{r/N \,:\, |r/N - x| \leq 4/N \text{ for some } x \in G\}.$$

By construction,

$$\sigma(G, F) = \sup_{\mathbf{y} \in G} d(y, F) \leq 1/N \text{ and } \sigma(F, G) = \sup_{\mathbf{y} \in F} d(y, G) \leq 4/N$$

so

$$\rho(G, F) = \sigma(G, F) + \sigma(F, G) \leq 5/N < \epsilon$$

But, if $y \in G$, then either $y + 1/N$ or $y - 1/N$ (or both) lies in G, so $G \notin \mathcal{E}_k$.

(iii) Observe that $\mathcal{E} = \cup_{k=1}^{\infty} \mathcal{E}_k$. $\square$

Proof of Lemma 13.12. (i) Suppose that $F_n \in \mathcal{F}_{j,k}$ and $F_n \to F$ in the Hausdorff metric. By definition, $F_n \supseteq [j/k, (j+1)/k]$, so $F \supseteq [j/k, (j+1)/k]$.

Thus $\mathcal{F}_{j,k}$ is closed.

(ii) Let $G \in \mathcal{K}$ and let $\epsilon > 0$. Choose an integer $N > 2\epsilon^{-1} + 1$. Let

$$E = \{r/N \,:\, |r/N - x| \leq 1/N \text{ for some } x \in G\}.$$

By construction,

$$\sigma(G, E) = \sup_{\mathbf{y} \in G} d(y, E) \leq 1/N \text{ and } \sigma(E, G) = \sup_{\mathbf{y} \in E} d(y, G) \leq 1/N$$

so

$$\rho(G, E) = \sigma(G, E) + \sigma(E, G) \leq 2/N < \epsilon$$

but $E \notin \mathcal{F}_{j,k}$.

(iii) Observe that $\mathcal{F} = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k} \mathcal{F}_{j,k}$, so $\mathcal{F}$ is the countable union of closed nowhere dense sets. $\square$

Proof of Theorem 13.13. With the notation of Lemmas 13.11 and 13.12,

$$\mathcal{K} \setminus \mathcal{C} = \mathcal{E} \cup \mathcal{F}.$$

Since the union of two first category sets is of first category, $\mathcal{K} \setminus \mathcal{C}$ is of the first category. □

Solution of Exercise 13.14. (i) (This may be familiar from 1B.) Set

$$g_n(x) = \begin{cases} 2^{2n}x & \text{if } 0 \leq x \leq 2^{-n-1}, \\ 2^{2n}(2^{-n} - x) & \text{if } 2^{-n-1} \leq x \leq 2^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

If $x \neq 0$, then $x \geq 2^{-m}$ for some m and so $g_n(x) = 0$ for $n \geq m$. Since $g_n(0) = 0$ for all n, we have $g_n(x) \to 0$ as $n \to \infty$ for all x.

However,

$$\sup_{t \in [0,1]} g_n(t) = g(2^{-n-1}) = 2^{n-1} \to \infty$$

as $n \to \infty$.

(ii) Extend g_n to a function on $\mathbb{R}$ by setting $g_n(t) = 0$ for $t \notin [0,1]$. Set $f_n(t) = \sum_{j=1}^{\infty} 2^{-j} g_n\big(2^j(t - 2^{-j})\big)$ and use (i). □

Proof of Theorem 13.15. Observe that

$$E_{n,m} = \{x : |f_n(x)| \leq m\} = f_n^{-1}([-m, m])$$

is closed (since f_n is continuous), so

$$E_m = \bigcap_{n=1}^{\infty} E_{n,m}$$

is.

If we fix $x \in [0,1]$ for the moment, we know that $f_n(x) \to 0$ as $n \to \infty$. In particular, we can find an $N(x)$ such that $|f_n(x)| \leq 1$ for all $n \geq N(x)$. Thus

$$|f_n(x)| \leq \max\{1, \max_{1 \leq j \leq N(x)} |f_n(x)|\}$$

and so $x \in E_{m(x)}$ for some integer $m(x)$.

The previous paragraph shows that

$$[0,1] = \bigcup_{m=1}^{\infty} E_m,$$

but Baire's category theorem tells us that $[0,1]$ cannot be the countable union of closed sets with empty interior. Thus there must exist an M such that E_M has non-empty interior, so $E_M \supseteq (a,b)$ for some non-empty interval (a,b). □

Solution of Exercise 14.1.

$$\begin{aligned}\frac{100}{37} &= 2 + \frac{26}{37},\\ \frac{37}{26} &= 1 + \frac{11}{26},\\ \frac{26}{11} &= 2 + \frac{4}{11},\\ \frac{11}{4} &= 2 + \frac{3}{4},\\ \frac{4}{3} &= 1 + \frac{1}{3}.\end{aligned}$$

Thus

$$\frac{100}{37} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}}}.$$

□

Lemma 14.2. (i) This is immediate.

(ii) We saw in 1A that the Euclidean algorithm terminates. (Or we could repeat the 1A proof by observing that the elements of the pairs are strictly decreasing.) □

Solution of Exercise 14.3. We know that $1 < \sqrt{2} < 2$, so

$$\sqrt{2} = 1 + \alpha$$

with $0 < \alpha = \sqrt{2} - 1 < 1$.

Now

$$\frac{1}{\alpha} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 = 2 + \alpha$$

so $N(\alpha) = 2$ and $T(\alpha) = 2 + \alpha$. Thus $\sqrt{2}$ has the *non-terminating* continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}$$

and cannot be rational. □

Solution of Exercise 14.4. We have

$$Dx = 10x - [10x] = 10x - Nx$$

so

$$x = 10^{-1}(Dx + Nx).$$

Since $Dx \in [0, 1)$, we have

$$\begin{aligned} x &= 10^{-1}(Dx + Nx) = 10^{-1}\big(10^{-1}\big(D(Dx) + N(Dx)\big) + Nx\big) \\ &= 10^{-1}Nx + 10^{-2}NDx + 10^{-2}D^2x \\ &= 10^{-1}Nx + 10^{-2}NDx + 10^{-3}ND^2x + 10^{-3}D^3x \\ &= \dots. \end{aligned}$$

We have

$$\begin{aligned} &\Pr(ND^rX = k_r \text{ for } 1 \leq r \leq n) \\ &\qquad\qquad = \Pr\left(\sum_{r=1}^{n} k_r 10^{-r} \leq X < 10^{-n} + \sum_{r=1}^{n} k_r 10^{-r}\right) = 10^{-n}, \end{aligned}$$

so

$$\Pr(ND^kX = j) = 1/10$$

for $0 \leq j \leq 9$ and

$$\Pr(ND^rX = k_r \text{ for } 1 \leq r \leq n\} = 10^{-n} = \prod_{r=1}^{n} \Pr(ND^rX = k_r),$$

showing that NX, NDX, ND^2X, … are independent □

Proof of Lemma 14.5. Observe that

$$\begin{aligned}
\Pr(TX \leq a) &= \Pr(n \leq X^{-1} \leq n + a \text{ for some integer } n \geq 1) \\
&= \Pr\big((n+a)^{-1} \leq X \leq n^{-1} \text{ for some integer } n \geq 1\big) \\
&= \sum_{n=1}^{\infty} \Pr\big((n+a)^{-1} \leq X \leq n^{-1}\big) \\
&= \sum_{n=1}^{\infty} \frac{1}{\log 2} \int_{(n+a)^{-1}}^{n^{-1}} \frac{1}{1+x}\, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{\log 2} \Big[\log(1+x)\Big]_{(n+a)^{-1}}^{n^{-1}} \\
&= \frac{1}{\log 2} \sum_{n=1}^{\infty} \big(\log(1+n^{-1}) - \log(1+(n+a)^{-1})\big) \\
&= \frac{1}{\log 2} \sum_{n=1}^{\infty} \Big(\big(\log(n+1) - \log n\big) - \big(\log(1+n+a) - \log(n+a)\big)\Big) \\
&= \frac{1}{\log 2} \lim_{N\to\infty} \big(\log(N+1) - \log(1+N+a) + \log(1+a)\big) \\
&= \frac{1}{\log 2} \left(\log(1+a) + \lim_{N\to\infty} \log \frac{N+1}{1+N+a}\right) \\
&= \frac{\log(1+a)}{\log 2} = \Pr(X \leq a).
\end{aligned}$$

□

Proof of Corollary 14.6. By Lemma 14.5,

$$\begin{aligned}
\Pr(NT^m X = j) &= \Pr(NX = j) \\
&= \frac{1}{\log 2} \int_{j^{-1}}^{(j+1)^{-1}} \frac{1}{1+x}\, dx \\
&= \frac{1}{\log 2} \Big[\log(1+x)\Big]_{(j+1)^{-1}}^{j^{-1}} \\
&= \frac{1}{\log 2} \big(\log \frac{j+1}{j} - \log \frac{j+2}{j+1}\big) = \frac{1}{\log 2} \log \frac{(j+1)^2}{j(j+2)}.
\end{aligned}$$

□

Proof of Lemma 15.2. (i) We use backwards induction on k. Since

$$\frac{r_n}{s_n} = a_n$$

the result is true for $k = n$.

Suppose the result is true for $m+1$ with $0 \le m \le n-1$. Then, by definition,

$$\begin{pmatrix} r_m \\ s_m \end{pmatrix} = \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{m+1} \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} a_m r_{m+1} + s_{m+1} \\ r_{m+1} \end{pmatrix},$$

and, by the inductive hypothesis,

$$\begin{aligned} a_m + & \cfrac{1}{a_{m+1} + \cfrac{1}{a_{m+2} + \cfrac{1}{a_{m+3} + \cfrac{1}{a_{m+4} + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}} \\ &= a_m + \frac{1}{r_{m+1}/s_{m+1}} \\ &= a_m + \frac{s_{m+1}}{r_{m+1}} \\ &= \frac{a_m r_{m+1} + s_{m+1}}{r_{m+1}} = \frac{r_m}{s_m}. \end{aligned}$$

The required result now follows.

(ii) Apply (i) repeatedly. □

Proof of Lemma 15.3. (i) This is just a restatement of Lemma 15.2 (ii).

(ii) We have

$$\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_{n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

□

Proof of Theorem 15.4. (i) Using Lemma 15.3 (ii), we have

$$\begin{aligned}
p_k q_{k-1} - q_k p_{k-1} &= \det \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} \\
&= \det \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \\
&= \det \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \det \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \det \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \\
&= (-1)^{k+1}.
\end{aligned}$$

(ii) Either use the matricial formula

$$\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$$

or direct computation.

(iii) Follows from the formula of (i).

(iv) By (ii) (or direct observation), the q_k form a strictly increasing sequence of strictly positive integers. Thus the $q_{k-1}q_k$ form a strictly increasing sequence of strictly positive integers.

The formula of (i) gives

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = (-1)^{k+1} \frac{1}{q_k q_{k-1}},$$

so the remark of the previous paragraph shows that

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}$$

is an alternating sequence with decreasing magnitude. Thus

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k-2}}{q_{2k-2}}, \ \frac{p_{2k-1}}{q_{2k-1}} < \frac{p_{2k+1}}{q_{2k+1}}.$$

We also have

$$\left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right| = \frac{1}{q_k q_{k-1}} \to 0.$$

(v) A decreasing sequence bounded below tends to a limit, so

$$\frac{p_{2k+1}}{q_{2k+1}} \to \alpha$$

as $k \to \infty$ for some α. Since

$$\left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right| \to 0$$

this tells us that

$$\frac{p_{2k}}{q_{2k}} \to \alpha$$

as $k \to \infty$. Thus

$$\frac{p_n}{q_n} \to \alpha$$

and

$$\frac{p_{2k+1}}{q_{2k+1}} > \alpha > \frac{p_{2k}}{q_{2k}}.$$

□

Solution of Exercise 15.5. Observe that if $a > 0$

$$s > t > 0 \Rightarrow \frac{1}{a+t} > \frac{1}{a+s}.$$

Thus (using a formal induction if more details are required)

$$\frac{p_{2k}}{q_{2k}} < x < \frac{p_{2k-1}}{q_{2k-1}}.$$

We know from Theorem 15.4 (v) that

$$\frac{p_n}{q_n} \to \alpha,$$

so $\alpha = x$. □

Proof of Theorem 15.6. Observe that if q and u are positive integers with $q \leq q_n$, then

$$\left| \frac{u}{q} - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{q q_{n+1}},$$

with equality only if $q = q_n$ and, in this case, only if $u = p_n$. Thus p_n/q_n is the closest fraction of the form u/q (with $q \leq q_n$) to p_{n+1}/q_{n+1}. But α lies between p_n/q_n and p_{n+1}/q_{n+1}, so p_n/q_n is also the closest fraction of the form u/q (with $q \leq q_n$) to α. □

Proof of Theorem 15.7. We may assume that $0 < x < 1$ without loss of generality. Using the notation of this section we observe that x lies between p_n/q_n and p_{n+1}/q_{n+1} so

$$\left| \frac{p_n}{q_n} - x \right| < \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

□

Solution of Exercise 15.8. Theorem 12.7 with $n = 2$. □

Solution of Exercise 15.9. (i) Observe that

$$\sigma = \frac{1}{1+\sigma}$$

so that

$$\sigma^2 + \sigma - 1 = 0$$

and

$$\sigma = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $\sigma > 0$, we must have

$$\sigma = \frac{-1 + \sqrt{5}}{2}.$$

(ii) By Theorem 15.4 (iii), $q_k = a_k q_{k-1} + q_{k-2}$ and $p_k = a_k p_{k-1} + p_{k-2}$ with $a_k = 1$. Thus

$$q_k = q_{k-1} + q_{k-2} \text{ and } p_k = p_{k-1} + p_{k-2}.$$

Now $q_0 = 1 = F_1$, $q_1 = 1 = F_2$, $p_0 = 0 = F_0$, $p_1 = 1 = F_1$, so by an inductive argument (or general knowledge of recurrence relations),

$$q_n = F_{n+1} \text{ and } p_n = F_n.$$

(iii) By Theorem 15.4 (i),

$$F_{n+1}F_{n-1} - F_n^2 = -(p_{n-1}q_n - q_{n-1}p_n) = (-1)^{n+1}.$$

□

Solution of Exercise 15.10. By Theorem 15.6 F_n/F_{n+1} is closer to σ than any other fraction with denominator no larger than F_{n+1}. Thus

$$\begin{aligned}\left|\frac{p}{q} - \sigma\right| &\geq \max\left\{\left|\frac{F_n}{F_{n+1}} - \sigma\right|, \left|\frac{F_{n+1}}{F_{n+2}} - \sigma\right|\right\} \geq \frac{1}{2}\left(\left|\frac{F_n}{F_{n+1}} - \sigma\right| + \left|\frac{F_{n+1}}{F_{n+2}} - \sigma\right|\right)\\ &= \frac{1}{2}\left|\frac{F_n}{F_{n+1}} - \frac{F_{n+1}}{F_{n+2}}\right| = \frac{1}{2F_{n+1}F_{n+2}}\end{aligned}$$

whenever $q \leq F_n$ and so whenever $F_{n-1} \leq q \leq F_n$.

Now $F_r \leq 2F_{r-1}$ so

$$\left|\frac{p}{q} - \sigma\right| \geq \frac{1}{2F_{n+1}F_{n+2}} \geq \frac{1}{64F_{n-1}^2} \geq \frac{1}{64q^2}$$

whenever $F_{n-1} \leq q \leq F_n$ for all $n \geq 2$ and the result follows. □

Solution of Exercise 15.11. Observe that $F_5 = 5,\ F_6 = 8,\ F_7 = 13$ so the difference in areas is 1 so we only need to hide one unit of area.

Identify the Fibonacci numbers in the diagram and generalise. $\square$

Solution of Exercise 16.2. (i) The proof follows the lines of that of Lemma 15.2.

Observe that, if s, $r \neq 0$,

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} ar + bs \\ r \end{pmatrix},$$

and

$$\frac{ar + bs}{r} = a + \frac{b}{r/s}.$$

Thus, by induction, if

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_{n-1} & b_{n-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix},$$

then

$$\frac{p_n}{q_n} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{\ddots \; \overline{\;a_{n-1} + \cfrac{b_{n-1}}{a_n}\;}}}}}}.$$

Observe that

$$\begin{aligned} &\begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_{n-1} & b_{n-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\qquad = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_{n-2} & b_{n-2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \end{aligned}$$

and thus

$$\begin{pmatrix} p_n & b_n p_{n-1} \\ q_n & b_n q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}.$$

We deduce that

$$\begin{pmatrix} p_n & b_n p_{n-1} \\ q_n & b_n q_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & b_{n-1} p_{n-2} \\ q_{n-1} & b_{n-1} q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}.$$

Looking at the first column gives

$$\begin{aligned} p_n &= a_n p_{n-1} + b_{n-1} p_{n-2}, \\ q_n &= a_n q_{n-1} + b_{n-1} q_{n-2}, \end{aligned}$$

as required. □

Proof of Lemma 16.3. We have

$$S_0(x) = \int_0^x \cos t \, dt = \sin x,$$

so $p_0(x) = 0$, $q_0(x) = 1$. Integration by parts gives

$$\begin{aligned} S_1(x) &= \frac{1}{2} \int_0^x (x^2 - t^2) \cos t \, dt \\ &= \left[\frac{1}{2}(x^2 - t^2) \sin t \right]_0^x + \int_0^x t \sin t \, dt = \int_0^x t \sin t \, dt \\ &= [-t \cos t]_0^x + \int_0^x t \cos t \, dt = -x \cos x + \sin x \end{aligned}$$

so $p_1(x) = x$, $q_1(x) = 1$.

If $n \geq 2$, a similar repeated integration by parts gives

$$\begin{aligned} S_n(x) &= \frac{1}{2^n n!} \int_0^x (x^2 - t^2)^n \cos t \, dt \\ &= \left[\frac{1}{2^n n!} (x^2 - t^2)^n \sin t \right]_0^x + \frac{1}{2^{n-1}(n-1)!} \int_0^x t(x^2 - t^2)^{n-1} \sin t \, dt \\ &= \frac{1}{2^{n-1}(n-1)!} \int_0^x t(x^2 - t^2)^{n-1} \sin t \, dt \\ &= \frac{1}{2^{n-1}(n-1)!} \left[-t(x^2 - t^2)^{n-1} \cos t \right]_0^x \\ &\quad + \frac{1}{2^{n-1}(n-1)!} \int_0^x \left((x^2 - t^2)^{n-1} - 2(n-1)t^2(x^2 - t^2)^{n-2} \right) \cos t \, dt \\ &= S_{n-1}(x) - \frac{1}{2^{n-2}(n-2)!} \int_0^x \left(t^2(x^2 - t^2)^{n-2} \right) \cos t \, dt \\ &= S_{n-1}(x) + \frac{1}{2^{n-2}(n-2)!} \left(\int_0^x \left((x^2 - t^2) t^2 (x^2 - t^2)^{n-2} \right) \cos t \, dt \right. \\ &\quad \left. - x^2 \int_0^x (x^2 - t^2)^{n-2} \cos t \, dt \right) \\ &= (2n-1) S_{n-1}(x) - x^2 S_{n-2}(x). \end{aligned}$$

Thus

$$p_n(x) = (2n-1)p_{n-1}(x) - x^2 p_{n-2}(x),$$
$$q_n(x) = (2n-1)q_{n-1}(x) - x^2 q_{n-2}(x)$$

and we are done. □

Proof of Theorem 16.1. Since

$$S_n(x) = q_n(x)\sin x - p_n(x)\cos x,$$

rearrangement gives

$$\tan x = \frac{p_n(x)}{q_n(x)} + \frac{S_n(x)}{q_n(x)\cos x}$$

so we need to show that

$$\frac{S_n(x)}{q_n(x)\cos x} \to 0.$$

It is easy to see that

$$\begin{aligned}|S_n(x)| &\leq \left|\frac{1}{2^n n!}\int_0^x (x^2-t^2)^n \cos t\,dt\right| \\ &\leq \frac{1}{2^n n!}|x||x|^{2n} \to 0\end{aligned}$$

as $n \to \infty$ for all x.

We shall show that if $|x| \leq 1$ then $q_n(x) \to \infty$. (Actually it can be shown that $|q_n(x)| \to \infty$ for all x.) Observe that

$$q_n(x) = (2n-1)q_{n-1}(x) - x^2 q_{n-2}(x)$$

so, if $|x| \leq 1$,

$$q_n(x) \geq (2n-1)q_{n-1}(x) - q_{n-2}(x) \geq 3q_{n-1}(x) - q_{n-2}(x)$$

for $n \geq 2$. Since $q_0(x) = q_1(x) = 1$ we have $q_2(x) \geq 2$ and a simple induction gives $q_n(x) \geq 2^{n-1}$ for $n \geq 1$ and $|x| \leq 1$.

Notice the rapidity of convergence of the continued fraction in this case. □

Proof of Theorem 17.1. We leave uniqueness to follow from Exercise 17.2. Let E be the set of $u \in [0,1]$ for which there exists a continuous function $\theta : [0,u] \to \mathbb{R}$ with $\theta(0) = \theta_0$ such that $g(t) = e^{i\theta(t)}$ for all $t \in [0,u]$. Since

$0 \in E$ (just take $\theta(0) = \theta_0$) and E is bounded, E must have an upper bound w.

Suppose that $w \in (0,1)$. Since g is continuous, we can find a $\delta > 0$ such that $(w-2\delta, w+2\delta) \subseteq [0,1]$ and $|g(t) - g(w)| < 1/2$ for $t \in (w-2\delta, w+2\delta)$ We know that there is a unique continuous function

$$\phi : \{z\,:\,|z-1| < 1/2,\ |z| = 1\} \to [-\pi/2, \pi/2]$$

such that

$$z = e^{i\phi(z)} \text{ for all } |z-1| < 1/2,\ |z| = 1.$$

Thus, if we choose θ_1 such that $e^{i\theta_1} = g(w)$ and define $\tilde{\theta} : (w-2\delta, w+2\delta) \to [-\pi/2, \pi/2]$ by

$$\tilde{\theta}(t) = \theta_1 + \phi\big(g(t)/g(w)\big)$$

we will have $\tilde{\theta}$ continuous and

$$g(t) = e^{i\tilde{\theta}(t)}$$

for $t \in (w-2\delta, w+2\delta)$.

By the definition of an upper bound, we can find $u \in (w-\delta, w]$ and a continuous function $\psi : [0,u] \to \mathbb{R}$ with $\psi(0) = \theta_0$ such that $g(t) = e^{i\psi(t)}$ for all $t \in [0,u]$. Since

$$e^{i\tilde{\theta}(u)} = g(u) = e^{i\psi(u)}$$

we must have $\tilde{\theta}(u) = \psi(u) + 2N\pi$ for some integer N. Taking

$$\theta(t) = \begin{cases} \psi(t) & \text{for } t \in [0,u] \\ \tilde{\theta}(t) - 2N\pi & \text{for } t \in [u, u+\delta] \end{cases}$$

we see that $\theta : [0, u+\delta] \to \mathbb{R}$ is a continuous function with $\theta(0) = \theta_0$ such that $g(t) = e^{i\theta(t)}$ for all $t \in [0, u+\delta]$. Thus $u+\delta \in E$ and $u + \delta > w$, contradicting our assumption that u is an upper bound.

A similar argument shows that 0 is not an upper bound. Thus $\sup E = 1$ and much the same argument as above shows that $1 \in E$, so we are done. $\square$

Solution of Exercise 17.2. Observe that $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) = \frac{\psi(t) - \phi(t)}{2\pi}$$

is an integer valued continuous function on $[0,1]$ and so must be constant. (Or quote the intermediate value theorem directly.) $\square$

Proof of Corollary 17.3. Set $g(t) = \gamma(t)/|\gamma(t)|$ and use Theorem 17.1. $\square$

Solution of Exercise 17.5. (i) Since $\gamma(0) \neq 0$ and

$$|\gamma(0)|e^{i\theta(0)} = \gamma(0) = \gamma(1) = |\gamma(1)|e^{i\theta(1)} = |\gamma(0)|e^{i\theta(1)},$$

we have $e^{i(\theta(1)-\theta(0))} = 1$, so $\theta(1) - \theta(0)$ is an integer multiple of 2π.

(ii) If we take $\gamma(t) = \exp(irt)$, we get $w(\gamma, 0) = r$ $[r \in \mathbb{Z}]$ □

Proof of Lemma 17.6. By Corollary 17.3, we can write

$$\gamma_j(t) = |\gamma_j(t)| \exp\big(i\theta_j(t)\big)$$

with $\theta_j : [0, 1] \to \mathbb{R}$ continuous. We now have

$$\begin{aligned}\gamma_1(t)\gamma_2(t) &= |\gamma_1(t)| \exp\big(i\theta_1(t)\big)|\gamma_2(t)| \exp\big(i\theta_2(t)\big)\\ &= |\gamma_1(t)\gamma_2(t)| \exp\big(i(\theta_1(t) + \theta_2(t))\big),\end{aligned}$$

so that

$$\begin{aligned}w(\gamma_1\gamma_2, 0) &= \frac{1}{2\pi}\Big(\big(\theta_1(1) + \theta_2(1)\big) - \big(\theta_1(0) + \theta_2(0)\big)\\ &= \frac{1}{2\pi}\big(\theta_1(1) - \theta_1(0)\big) + \frac{1}{2\pi}\big(\theta_2(1) - \theta_2(0)\big) = w(\gamma_1, 0) + w(\gamma_2, 0).\end{aligned}$$

□

Proof of Lemma 17.7. This argument may be familiar from 1B complex variable.

Write $\gamma(t) = \big(1 + \gamma_2(t)/\gamma_1(t)\big)$. By Lemma 17.6,

$$w(\gamma_1 + \gamma_2, 0) = w(\gamma_1\gamma, 0) = w(\gamma_1, 0) + w(\gamma, 0),$$

so it suffices to prove that $w(\gamma, 0) = 0$. We shall do this by noting that $|\gamma_2(t)/\gamma_1(t)| < 1$ and so

$$\Re\gamma(t) > 0$$

for all $t \in [0, 1]$.

By Corollary 17.3, we can write

$$\gamma(t) = |\gamma(t)| \exp\big(i\theta(t)\big)$$

with $\theta : [0, 1] \to \mathbb{R}$ continuous and $\theta(0) \in (-\pi/2, \pi/2)$. If $|\theta(t)| \geq \pi/2$ for any $t \in [0, 1]$, the intermediate value theorem tells us that there is an $s \in [0, t]$ such that $|\theta(s)| = \pi/2$ and so $\Re\gamma(s) = 0$, which is impossible. Thus $|\theta(t)| < \pi/2$ for all $t \in [0, 1]$.

In particular $|\theta(0)|, |\theta(1)| < \pi/2$, so $|\theta(1) - \theta(0)| < \pi$. It follows that $w(\gamma, 0)$ is an integer with $|w(\gamma, 0)| < 1/2$ and so $w(\gamma, 0) = 0$. □

Solution of Exercise 17.9. Setting

$$\Gamma(s,t) = \gamma(t),$$

we see that $\gamma \simeq \gamma$.

If $\gamma_0 \simeq \gamma_1$, then we can find a continuous function $\Gamma : [0,1]^2 \to \mathbb{C} \setminus \{0\}$ such that

$$\begin{aligned} \Gamma(s,0) &= \Gamma(s,1) && \text{for all } s \in [0,1], \\ \Gamma(0,t) &= \gamma_0(t) && \text{for all } t \in [0,1], \\ \Gamma(1,t) &= \gamma_1(t) && \text{for all } t \in [0,1]. \end{aligned}$$

If we set $\tilde{\Gamma}(s,t) = \Gamma(1-s,t)$, then $\tilde{\Gamma} : [0,1]^2 \to \mathbb{C} \setminus \{0\}$ is a continuous function such that

$$\begin{aligned} \tilde{\Gamma}(s,0) &= \tilde{\Gamma}(s,1) && \text{for all } s \in [0,1], \\ \tilde{\Gamma}(0,t) &= \gamma_1(t) && \text{for all } t \in [0,1], \\ \tilde{\Gamma}(1,t) &= \gamma_0(t) && \text{for all } t \in [0,1], \end{aligned}$$

and so $\gamma_1 \simeq \gamma_0$.

If $\gamma_0 \simeq \gamma_1$ and $\gamma_1 \simeq \gamma_2$, then we can find a continuous functions $\Gamma_j : [0,1]^2 \to \mathbb{C} \setminus \{0\}$ such that

$$\begin{aligned} \Gamma_j(s,0) &= \Gamma_j(s,1) && \text{for all } s \in [0,1], \\ \Gamma_j(0,t) &= \gamma_{0+j}(t) && \text{for all } t \in [0,1], \\ \Gamma_j(1,t) &= \gamma_{1+j}(t) && \text{for all } t \in [0,1] \end{aligned}$$

for $j = 0$, 1.

If we set

$$\Gamma(s,t) = \begin{cases} \Gamma_0(2s,t) & \text{for all } s \in [0,1/2],\ t \in [0,1] \\ \Gamma_1(2s-1,t) & \text{for all } s \in (1/2,1],\ t \in [0,1] \end{cases}$$

then $\Gamma : [0,1]^2 \to \mathbb{C} \setminus \{0\}$ is a continuous function such that

$$\begin{aligned} \tilde{\Gamma}(s,0) &= \tilde{\Gamma}(s,1) && \text{for all } s \in [0,1], \\ \tilde{\Gamma}(0,t) &= \gamma_0(t) && \text{for all } t \in [0,1], \\ \tilde{\Gamma}(1,t) &= \gamma_2(t) && \text{for all } t \in [0,1], \end{aligned}$$

and so $\gamma_0 \simeq \gamma_2$. $\square$

Proof of Theorem 17.10. Let Γ be as in Definition 17.8. The map $(s,t) \mapsto |\Gamma(s,t)|$ is continuous so, by compactness, $|\Gamma(s,t)|$ attains a minimum m on the compact set $[0,1]^2$. Since Γ is never zero, we must have $m > 0$.

By compactness (see Theorem 7.7 if necessary), Γ is uniformly continuous and so we can find a strictly positive integer N such that

$$|s - s'|, |t - t'| < 2/N \Rightarrow |\Gamma(s,t) - \Gamma(s',t')| < m/2.$$

If $0 \leq r \leq N$ let us define

$$\beta_r(t) = \Gamma(r/N, t)$$

for $t \in [0,1]$. We observe that

$$\begin{aligned} |\beta_r(t)| &= |\Gamma(r/N,t)| \geq m > m/2 \\ &> \big|\Gamma\big(r/N,t\big) - \Gamma\big((r+1)/N,t\big)\big| = |\beta_r(t) - \beta_{r+1}(t)| \end{aligned}$$

for all $t \in [0,1]$, so by the dog walking lemma (Lemma 17.7),

$$w(\beta_r, 0) = w(\beta_{r+1}, 0)$$

for all $0 \leq r \leq N-1$. It follows that

$$w(\gamma_0, 0) = w(\beta_0, 0) = w(\beta_N, 0) = w(\gamma_1, 0).$$

□

Proof of Corollary 17.11. Suppose, if possible, that $f(z) \neq 0$ for $z \in D$. The nowhere-zero function $G : [0,1]^2 \mapsto \mathbb{C}$ given by

$$G(s,t) = f(se^{2\pi i t})$$

is continuous with $G(s,0) = G(s,1)$ for all $s \in [0,1]$, $G(1,t) = \gamma(t)$ and $G(0,t) = \gamma_0(t)$ where $\gamma_0(t) = f(0)$ for all $t \in [0,1]$. Thus γ and γ_0 are homotopic using closed curves not passing through 0. By Theorem 17.10, $w(\gamma, 0) = w(\gamma_0, 0) = 0$ contradicting our hypothesis. □

Proof of Corollary 17.12. It is sufficient to consider polynomials P of the form

$$P(z) = z^n + Q(z)$$

with $Q(z) = \sum_{j=0}^{n-1} a_j z^j$. If we set $R = 1 + \sum_{j=0}^{n-1} |a_j|$ and consider $p(z) = R^{-n}p(Rz)$ we see that P has a root if p has root and that

$$p(z) = z^n + q(z)$$

with $|q(z)| < 1$ when $|z| = 1$.

By the dog walking lemma, the map $t \mapsto p(e^{2\pi i t})$ for $t \in [0,1]$ has the same winding number as $t \mapsto (e^{2\pi i t})^n = e^{2\pi i n t}$, that is to say, n. By Corollary 17.11, there must exist a $z \in D$ with $p(z) = 0$, so we are done. □

Proof of Corollary 17.13. Suppose such a function existed. The continuous map $G : [0,1]^2 \to \partial D$ given by

$$G(s,t) = f(se^{2\pi it})$$

gives a homotopy between γ_0 defined by $\gamma_0(t) = f(0)$ and γ_1 defined by $\gamma_1(t) = f(t) = e^{2\pi t}$ using closed curves not passing through 0. By Theorem 17.10, this gives

$$1 = w(\gamma_1, 0) = w(\gamma_0, 0) = 0,$$

which is absurd.

The required result follows by reductio ad absurdum. □

'You are old, Father William' the young man said,
And your hair has become very white;
And yet you incessantly stand on your head
Do you think, at your age, it is right?'

'In my youth' Father William replied to his son,
'I feared it might injure the brain;
But, now that I'm perfectly sure I have none,
Why, I do it again and again.'

'You are old' said the youth 'as I mentioned before,
And have grown most uncommonly fat;
Yet you turned a back-somersault in at the door –
Pray, what is the reason of that?'

'In my youth' said the sage, as he shook his grey locks,
'I kept all my limbs very supple
By the use of this ointment – one shilling the box –
Allow me to sell you a couple?'

'You are old' said the youth 'and your jaws are too weak
For anything tougher than suet;
Yet you finished the goose, with the bones and the beak –
Pray how did you manage to do it?'

'In my youth' said his father 'I took to the law,
And argued each case with my wife;
And the muscular strength, which it gave to my jaw,
Has lasted the rest of my life.'

'You are old' said the youth 'one would hardly suppose
That your eye was as steady as ever;
Yet you balanced an eel on the end of your nose –
What made you so awfully clever?'

'I have answered three questions, and that is enough'
Said his father; 'don't give yourself airs!
Do you think I can listen all day to such stuff?
Be off, or I'll kick you down stairs!'

Lewis Carroll