Topics in Analysis: Example Sheet 4

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(1) Let $f : \mathbf{C} \to \mathbf{C}$ be a continuous function. If there is a positive integer n and a non-zero complex number c such that

$$\lim_{z\to\infty} z^{-n} f(z) = c,$$

prove that $f(z_0) = 0$ for some $z_0 \in \mathbf{C}$.

(2) Let $\gamma : [0,1] \to \mathbf{C} \setminus \{0\}$ be a continuous map such that $\gamma(0) = \gamma(1)$ and suppose that for each $t \in [0,1]$, $\gamma(t)$ is not equal to a negative real number. By considering $\gamma(t) + c$ for suitable values of $c \in [0,\infty)$, or otherwise, show that the winding number $w(\gamma; 0) = 0$.

(3) Let $g : \mathbf{S}^1 \to \mathbf{S}^1$ be a continuous map, where $\mathbf{S}^1 = \{z \in \mathbf{C} : |z| = 1\}$. If there is a continuous extension of g to the closed unit disk $D = \{z \in \mathbf{C} : |z| \leq 1\}$ (i.e. if there is a continuous map $G : D \to \mathbf{S}^1$ such that $G(z) = g(z)$ for each $z \in \mathbf{S}^1$), prove that

(a) $g(z) = z$ for some $z \in \mathbf{S}^1$.

(b) $g(z) = -z$ for some $z \in \mathbf{S}^1$.

(4) Let $f : [1,\infty) \to \mathbf{R}$ be a continuous function and suppose that for each $x \in [1,\infty)$, $f(nx) \to 0$ as $n \to \infty$, $n \in \mathbf{N}$. Prove that $\lim_{x\to\infty} f(x) = 0$. (Hint: For $\epsilon > 0$, consider the sets $Q_k = \{x \in [1,\infty) : |f(nx)| < \epsilon \quad \forall n \geq k\}$.)

(5) Let K be a non-empty compact, connected subset of the complex plane. Let f be a complex function on K which is analytic in some open set containing K. Prove that either f is a polynomial on K or the nth derivative $f^{(n)}(z) \neq 0$ for some $z \in K$ and all $n = 1, 2, 3, \ldots$

(6) Let A_j be a sequence of subsets of $[0,1]$ such that for each $N \geq 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0,1]$. Prove that the set S of points $x \in [0,1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$?

(7) If G is an open dense subset of $\mathbf{R}$, and $\mathbf{Q}$ is the set of rationals, show that $G \setminus \mathbf{Q}$ must be dense in $\mathbf{R}$. If we only assume G is uncountable and dense in $\mathbf{R}$, does it still follow that $G \setminus \mathbf{Q}$ is dense in $\mathbf{R}$?

(8) Show that there is a dense set of real numbers x with the property that for each positive integer n, there exist integers p and q with $q \geq 2$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

(9) Prove that the set of (real) transcendental numbers is uncountable (i) by showing that the set of algebraic numbers is countable, and (ii) by considering the numbers $\sum_{n=0}^{\infty} \frac{b_n}{10^{n!}}$ with $b_n \in \{1, 2\}$.

(10) Determine the continued fraction expansions of $71/49$ and $\sqrt{3}$. Deduce that $\sqrt{3}$ is irrational.

(11) Let a, b be positive integers. Let

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \ldots}}}}.$$

(a) Show that x solves $ax^2 + abx - b = 0$.

(b) Assume $a = b = 1$. Show that, in this case, $x = \frac{-1+\sqrt{5}}{2}$ and that if $\frac{p_n}{q_n}$ is the nth convergent of the continued fraction, then $p_n = F_n$, $q_n = F_{n+1}$, where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

(c) Show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}$.

(12) Let p be a positive integer and α, β be real numbers such that $\alpha + \beta = \alpha\beta = -p$. Find the simple continued fraction representations of $|\alpha|$ and $|\beta|$ in terms of p.

(13) Determine the rational number with denominator ≤ 10 that best approximates $71/49$.

(14) Let $\alpha \in \mathbf{R}$ be irrational. Show that there are infinitely many rationals p/q ($p \in \mathbf{Z}$, $q \in \mathbf{N}$) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Show in fact that given any two consecutive convergents of α in their lowest form, at least one, p/q, satisfies this inequality.

(15) (a) For sequences of real numbers $\{a_n\}$ and $\{b_n\}$, consider

$$F = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}}.$$

Assuming at no stage we divide by zero and letting p_n/q_n be the nth convergent of this, derive, for appropriate choices of p_n, q_n, the relation

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ b_n & 0 \end{pmatrix}$$

and deduce that $p_{n+1} = a_{n+1}p_n + b_n p_{n-1}$, $q_{n+1} = a_{n+1}q_n + b_n q_{n-1}$ for $n \geq 1$, where $p_0 = a_0, q_0 = 1$, $p_1 = a_0 a_1 + b_0$ and $q_1 = a_1$.

(b) Use (a) to show that for $|x| \leq 1$,

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cdots}}}}.$$

(Hint: Let $s_n(x) = \frac{1}{2^n (n!)} \int_0^x (x^2 - t^2) \cos t \, dt$ and first prove the following two facts: (i) $s_n(x) = q_n(x) \sin x - p_n(x) \cos x$ where p_n, q_n are the polynomials defined by $p_0(x) = 0$, $p_1(x) = x$, $q_0(x) = 1$, $q_1(x) = 1$ and the relations $p_n(x) = (2n-1)p_{n-1}(x) - x^2 p_{n-2}(x)$, $q_n(x) = (2n-1)q_{n-1}(x) - x^2 q_{n-2}(x)$ for $n \geq 2$, (ii) $q_{n+1}(x) \geq q_n(x)$ and $q_n(x) \geq n!$ for $|x| \leq 1$ and $n = 0, 1, 2, \ldots$)

(c) Deduce that

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cdots}}}.$$