RIEMANN SURFACES EXAMPLES 1

Michaelmas 2017

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at hirriege@dpmms.cam.ac.uk.

1. Let \( U = \mathbb{C} \setminus ([-1,0] \cup [1,\infty]) \) and let \( \gamma \) be a closed curve in \( U \). Using standard properties of winding numbers, show that (i) \( n(\gamma, 1) = 0 \), and (ii) \( n(\gamma, 0) = n(\gamma, -1) \).

2. Let \( P(w_0, w_1, \ldots, w_s; z) \) be a polynomial in the \( s+1 \) complex variables \( w_0, w_1, \ldots, w_s \), where the coefficients of \( P \) are holomorphic on \( \mathbb{C} \). Thus
\[
P(f(z), f^{(1)}(z), \ldots, f^{(s)}(z); z) = 0
\]
is a differential equation, which we abbreviate to \( P(f) = 0 \). If \( (f, D) \) is a function element with \( P(f) = 0 \) in \( D \) and if \( (g, D') \approx (f, D) \) is an analytic continuation, then show that \( P(g) = 0 \) in \( D' \). Give an example of a differential equation and function elements as above, where \( D' = D \) but \( g \neq f \) on \( D \).

3. Let \( \pi : \hat{X} \to X \) be a covering map of topological spaces (recalling here that the spaces are assumed connected and Hausdorff), and \( f : \hat{X} \to \hat{X} \) a continuous map such that \( \pi f = \pi \). Show that \( f \) has no fixed points unless it is the identity.

4. Show that the power series \( f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n \) defines an analytic function \( (1-z) \log(1-z) + z \) on the unit disc \( D \). Deduce that the function element \( (f, D) \) defines a complete analytic function on \( \mathbb{C} \setminus \{1\} \), but does not extend to an analytic function on \( \mathbb{C} \setminus \{1\} \).

5. Show that the power series \( f(z) = \sum z^{2n}/2^n \) has the unit circle as a natural boundary.

6. Show that atlases being equivalent is an equivalence relation on the set of atlases. Show that any conformal structure on a Riemann surface contains a maximal atlas.

7. Let \( T \) be the complex torus \( \mathbb{C}/(1, \tau) \), and let \( Q_1 \subset \mathbb{C} \) be the open parallelogram with vertices \( 0, 1, \tau, 1+\tau \), and \( Q_2 \) the translation of \( Q_1 \) by \( (1+\tau)/2 \). Let \( U_1, U_2 \) denote the open subsets of \( T \) given by projection of \( Q_1, Q_2 \) respectively, and let \( \phi_1 : U_1 \to Q_1, \phi_2 : U_2 \to Q_2 \) be the charts obtained by taking the inverse maps. Describe explicitly the transition function
\[
\phi_2\phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2).
\]

8. By considering the singularity at \( \infty \) or otherwise, show that any injective analytic map \( f : \mathbb{C} \to \mathbb{C} \) has the form \( f(z) = az + b \), for some \( a \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). Find the injective analytic maps \( \mathbb{C}_\infty \to \mathbb{C}_\infty \).

9. Let \( \Lambda = \langle \tau_1, \tau_2 \rangle \) be a lattice in \( \mathbb{C} \) and let \( T = \mathbb{C}/\Lambda \) be the corresponding complex torus. Let \( \Lambda' \) denote the lattice \( (1, \tau_2/\tau_1) \) and \( T' = \mathbb{C}/\Lambda' \). Show that the Riemann surfaces \( T \) and \( T' \) are analytically isomorphic (i.e. conformally equivalent).

10. Define an equivalence relation \( \sim \) on \( \mathbb{C}^* \) by \( z \sim w \) iff \( z = 2^s w \) for some \( s \in \mathbb{Z} \). Show that the quotient space \( R = \mathbb{C}^*/\sim \) has the natural structure of a compact Riemann surface, and that \( R \) is analytically isomorphic to a complex torus.
11. (The identity principle for Riemann surfaces) Let $R, S$ be Riemann surfaces, and $f, g : R \to S$ be analytic maps between them. Set $E = \{ z \in R : f(z) = g(z) \}$; show that either $E = R$ or $E$ contains only isolated points.

12. Let $D \subset \mathbb{C}$ be an open disc and $u$ a harmonic function on $D$. Define a complex valued function $g$ on $D$ by $g(z) = u_x - iu_y$; show that $g$ is analytic. If $z_0$ denotes the centre of the disc, define a function $f$ on $D$ by $f(z) = u(0) + \int_{z_0}^z g$, the integral being taken over the straight line segment. Show that $f$ is analytic with $f' = g$, and that $u = \Re f$.

13. Suppose $u, v$ are harmonic functions on a Riemann surface $R$ and $E = \{ z \in R : u(z) = v(z) \}$. Show that either $E = R$, or $E$ has empty interior. Give an example to show that $E$ does not in general consist of isolated points.

14. Let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ both be sets of four distinct points in $\mathbb{C}_\infty$. Show that any analytic isomorphism $f : \mathbb{C}_\infty \setminus \{a_1, a_2, a_3, a_4\} \to \mathbb{C}_\infty \setminus \{b_1, b_2, b_3, b_4\}$ extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$. Using your answer to Question 8, find a necessary and sufficient condition for $\mathbb{C} \setminus \{0, 1, a\}$ to be conformally equivalent to $\mathbb{C} \setminus \{0, 1, b\}$, where $a, b$ are complex numbers distinct from 0 and 1.

15. Let $f(z)$ be the complex polynomial $z^3 - z$; consider the subspace $R$ of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ given by the equation $w^2 = f(z)$, where $(z, w)$ denote the coordinates on $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that $R$ has the structure of a Riemann surface, on which $\pi$ is an analytic map. If $g$ denotes the projection onto the second factor, show that $g$ is also an analytic map.

By deleting three appropriate points from $R$, show that $\pi$ yields a covering map from the resulting Riemann surface $R_0 \subset R$ to $\mathbb{C} \setminus \{-1, 0, 1\}$, and that $R_0$ is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function $(z^3 - z)^{1/2}$ over $\mathbb{C} \setminus \{-1, 0, 1\}$.

16. Let $f(z) = \sum a_n z^n$ be a power series of radius of convergence 1, and for $w$ in the open unit disc, set $\rho(w)$ to be the radius of convergence for the power series expansion about $w$ (so that $\rho(0) = 1$). Show that a point $\zeta \in C(0, 1)$ on the unit circle is regular if and only if $\rho(\zeta/2) > 1/2$. Suppose furthermore that all the $a_n$ are non-negative real numbers. If $\zeta \in C(0, 1)$, show that $|f^{(r)}(\zeta/2)| \leq f^{(r)}(1/2)$ for all $r$, and hence that $\rho(\zeta/2) \geq \rho(1/2)$. Deduce that 1 is a singular point.