Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at h.krieger@dpmms.cam.ac.uk.

1. Suppose $\Omega \subset \mathbb{C}$ is an additive subgroup such that $\Omega$ contains only isolated points. Show that either $\Omega = \{0\}$, or $\Omega = \mathbb{Z}\omega$ for some $\omega \neq 0$, or $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \neq 0$ and $\omega_2/\omega_1 \notin \mathbb{R}$.

2. Suppose that $f$ is a simply periodic analytic function on $\mathbb{C}$ with periods $\mathbb{Z}$, and that $\lim_{y \to +\infty} f(x + iy)$ and $\lim_{y \to -\infty} f(x + iy)$ both exist (possibly $\infty$) uniformly in $x$. Show that $f(z) = \sum_{k=-N}^{N} a_k e^{2\pi i k z}$, i.e. $f(z)$ has a finite Fourier expansion.

3. Let $f$ be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let $P \subset \mathbb{C}$ be a fundamental parallelogram; using the argument principle, and if necessary slightly perturbing $P$, show that the number of zeros of $f$ in $P$ is the same as the number of poles, both counted with multiplicities (in lectures, this followed by a use of the Valency theorem, but this more direct argument via contour integration also works).

4. With the notation as in the previous question, let the degree of $f$ be $n$, and let $a_1, \ldots, a_n$ denote the zeros of $f$ in a fundamental parallelogram $P$, and let $b_1, \ldots, b_n$ denote the poles (both with possible repeats). By considering the integral (if required, also slightly perturbing $P$)

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)} dz,$$

show that

$$\sum_{j=1}^{n} a_j - \sum_{j=1}^{n} b_j \in \Lambda.$$

5. Suppose $a$ is a complex number with $|a| > 1$. Show that any analytic function $f$ on $\mathbb{C}^*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}^*$ must be constant, but that there is a non-constant meromorphic function $f$ on $\mathbb{C}^*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}^*$.

6. Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp$ satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$, for some constant $A \in \mathbb{C}$. Show that there are at least three points and at most five points (modulo $\Lambda$) at which $\wp'$ is not locally injective.

7. With notation as in the previous question, and $a$ a complex number with $2a \notin \Lambda$, show that the elliptic function

$$h(z) = (\wp(z - a) - \wp(z + a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

has no poles on $\mathbb{C} \setminus \Lambda$. By considering the behaviour of $h$ at $z = 0$, deduce that $h$ is constant, and show that this constant is zero.

8. Find an explicit regular covering map of Riemann surfaces $\Delta \to \Delta^*$, where $\Delta$ here denotes the open unit disc and $\Delta^*$ the punctured disc.

9. Show that $\mathbb{C} \setminus \{P, Q\}$, where $P \neq Q$, is not conformally equivalent to $\mathbb{C}$ or $\mathbb{C}^*$, and deduce from the Uniformization theorem that it is uniformized by the open unit disc $\Delta$. Show that the same is true for any domain in $\mathbb{C}$ whose complement has more than one point.

10. Let $R$ be a compact Riemann surface of genus $g$ and $P_1, \ldots, P_n$ be distinct points of $R$. Show that $R \setminus \{P_1, \ldots, P_n\}$ is uniformized by the open unit disc $\Delta$ if and only if $2g - 2 + n > 0$, and by $\mathbb{C}$ if and only if $2g - 2 + n = 0$ or $-1$.

11. Let $f, g$ be meromorphic functions on a compact Riemann surface $R$. Show that there is a non-zero polynomial $P(w_1, w_2)$ such that $P(f, g) = 0$. 

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[Hint: Suppose \( f, g \) have valencies \( m, n \) respectively, and put \( d = m + n \). Show that it is possible to choose complex numbers \( a_{ij} \), not all zero, such that the function

\[
\sum_{j=0}^{d} \sum_{k=0}^{d} a_{jk} f(z)^j g(z)^k
\]

has at least \((d^2 + 2d)\) distinct zeros in \( R \). Show that it cannot have more than \( d^2 \) poles, and deduce that it must be identically zero on \( R \).

12. Prove from first principles that \( S^2 \) is simply connected (this is not quite as trivial as it initially looks).