RIEMANN SURFACES EXAMPLES 1

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be
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1. Let \( U = \mathbb{C} \setminus ([-1, 0] \cup [1, \infty]) \) and let \( \gamma \) be a closed curve in \( U \). Using standard properties of
winding numbers, show that (i) \( n(\gamma, 1) = 0 \), and (ii) \( n(\gamma, 0) = n(\gamma, -1) \).

2. Let \( P(w_0, w_1, \ldots, w_s; z) \) be a polynomial in the \( s+1 \) complex variables \( w_0, w_1, \ldots, w_s \), where the
coefficients of \( P \) are holomorphic on \( \mathbb{C} \). Thus
\[
P(f(z), f^{(1)}(z), \ldots, f^{(s)}(z); z) = 0
\]
is a differential equation, which we abbreviate to \( P(f) = 0 \). If \( (f, D) \) is a function element with
\( P(f) = 0 \) in \( D \) and if \( (g, D') \approx (f, D) \) is an analytic continuation, then show that \( P(g) = 0 \) in \( D' \).
Give an example of a differential equation and function elements as above, where \( D' = D \) but \( g \neq f \)
on \( D \).

3. Let \( \pi : \tilde{X} \to X \) be a covering map of topological spaces (recalling here that the spaces are
assumed connected and Hausdorff), and \( f : \tilde{X} \to \tilde{X} \) a continuous map such that \( \pi f = \pi \). Show that
\( f \) has no fixed points unless it is the identity.

4. Show that the power series \( f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n \) defines an analytic function \( (1-z) \log(1-z) + z \)
on the unit disc \( D \). Deduce that the function element \( (f, D) \) defines a complete analytic function on
\( \mathbb{C} \setminus \{1\} \), but does not extend to an analytic function on \( \mathbb{C} \setminus \{1\} \).

5. Show that the power series \( f(z) = \sum z^{2n}/2^n \) has the unit circle as a natural boundary.

6. Show that atlases being equivalent is an equivalence relation on the set of atlases. Show that any
conformal structure on a Riemann surface contains a maximal atlas.

7. Let \( T \) be the complex torus \( \mathbb{C}/\{1, \tau\} \), and let \( Q_1 \subset \mathbb{C} \) be the open parallelogram with vertices
\( 0, 1, \tau, 1+\tau \), and \( Q_2 \) the translation of \( Q_1 \) by \( (1+\tau)/2 \). Let \( U_1, U_2 \) denote the open subsets of \( T \) given
by projection of \( Q_1, Q_2 \) respectively, and let \( \phi_1 : U_1 \to Q_1, \phi_2 : U_2 \to Q_2 \) be the charts obtained by
taking the inverse maps. Describe explicitly the transition function
\[
\phi_2\phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2).
\]

8. By considering the singularity at \( \infty \) or otherwise, show that any injective analytic map \( f : \mathbb{C} \to \mathbb{C} \)
has the form \( f(z) = az + b \), for some \( a \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). Find the injective analytic maps \( \mathbb{C}_\infty \to \mathbb{C}_\infty \).

9. Let \( \Lambda = \langle \tau_1, \tau_2 \rangle \) be a lattice in \( \mathbb{C} \) and let \( T = \mathbb{C}/\Lambda \) be the corresponding complex torus. Let
\( \Lambda' \) denote the lattice \( \langle 1, \tau_2/\tau_1 \rangle \) and \( T' = \mathbb{C}/\Lambda' \). Show that the Riemann surfaces \( T \) and \( T' \) are
analytically isomorphic (i.e. conformally equivalent).

10. Define an equivalence relation \( \sim \) on \( \mathbb{C}^* \) by \( z \sim w \) iff \( z = 2^sw \) for some \( s \in \mathbb{Z} \). Show that the
quotient space \( R = \mathbb{C}^*/\sim \) has the natural structure of a compact Riemann surface, and that \( R \) is
analytically isomorphic to a complex torus.

11. (The identity principle for Riemann surfaces) Let \( R, S \) be Riemann surfaces, and \( f, g : R \to S \)
be analytic maps between them. Set \( E = \{z \in R : f(z) = g(z)\} \); show that either \( E = R \) or \( E \)
contains only isolated points.
12. Let \( D \subset \mathbb{C} \) be an open disc and \( u \) a harmonic function on \( D \). Define a complex valued function \( g \) on \( D \) by \( g = u_x - iu_y \); show that \( g \) is analytic. If \( z_0 \) denotes the centre of the disc, define a function \( f \) on \( D \) by
\[
 f(z) = u(z_0) + \int_{z_0}^{z} g, 
\]
the integral being taken over the straight line segment. Show that \( f \) is analytic with \( f' = g \), and that \( u = \Re f \).

13. Suppose \( u, v \) are harmonic functions on a Riemann surface \( R \) and \( E = \{ z \in R : u(z) = v(z) \} \).
Show that either \( E = R \), or \( E \) has empty interior. Give an example to show that \( E \) does not in general consist of isolated points.

14. Let \( \{ a_1, a_2, a_3, a_4 \} \) and \( \{ b_1, b_2, b_3, b_4 \} \) both be sets of four distinct points in \( \mathbb{C}_\infty \).
Show that any analytic isomorphism \( f : \mathbb{C}_\infty \setminus \{ a_1, a_2, a_3, a_4 \} \to \mathbb{C}_\infty \setminus \{ b_1, b_2, b_3, b_4 \} \)
extends to an analytic isomorphism \( \mathbb{C}_\infty \to \mathbb{C}_\infty \). Using your answer to Question 8, find a necessary and sufficient condition for \( \mathbb{C} \setminus \{ 0,1,a \} \) to be conformally equivalent to \( \mathbb{C} \setminus \{ 0,1,b \} \), where \( a, b \) are complex numbers distinct from 0 and 1.

15. Let \( f(z) \) be the complex polynomial \( z^3 - z \); consider the subspace \( R \) of \( \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} \) given by the equation \( w^2 = f(z) \), where \( (z,w) \) denote the coordinates on \( \mathbb{C}^2 \), and let \( \pi : R \to \mathbb{C} \) be the restriction of the projection map onto the first factor. Show that \( R \) has the structure of a Riemann surface, on which \( \pi \) is an analytic map. If \( g \) denotes the projection onto the second factor, show that \( g \) is also an analytic map.

By deleting three appropriate points from \( R \), show that \( \pi \) yields a covering map from the resulting Riemann surface \( R_0 \subset R \) to \( \mathbb{C} \setminus \{ -1,0,1 \} \), and that \( R_0 \) is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function \( (z^3 - z)^{1/2} \) over \( \mathbb{C} \setminus \{ -1,0,1 \} \).

16. Let \( f(z) = \sum a_n z^n \) be a power series of radius of convergence 1, and for \( w \) in the open unit disc, set \( \rho(w) \) to be the radius of convergence for the power series expansion about \( w \) (so that \( \rho(0) = 1 \)). Show that a point \( \zeta \in C(0,1) \) on the unit circle is regular if and only if \( \rho(\zeta/2) > 1/2 \).
Suppose furthermore that all the \( a_n \) are non-negative real numbers. If \( \zeta \in C(0,1) \), show that \( |f^{(r)}(\zeta/2)| \leq f^{(r)}(1/2) \) for all \( r \), and hence that \( \rho(\zeta/2) \geq \rho(1/2) \). Deduce that 1 is a singular point.