Example Sheet 2

(1) Let $U \subset \mathbb{C}$ be a star-domain; show that it is simply connected.

(2) Let $\pi : \tilde{X} \to X$ be a regular covering map of topological spaces; show that $\pi$ is surjective. Suppose now that $X$ is simply connected; using the Monodromy theorem, show that $\pi$ is a homeomorphism.

(3) Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an analytic map of complex tori, where $\pi_j$ denotes the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is an analytic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 F = f \pi_1$.

[HINT: Define $F$ as follows. Choose a point $\mu$ in $\mathbb{C}$ such that $\pi_2(\mu) = f \pi_1(0)$. For $z \in \mathbb{C}$, join 0 to $z$ by a path $\gamma : [0, 1] \to \mathbb{C}$, and observe that the path $f \pi_1 \gamma$ in $\mathbb{C}/\Lambda_2$ has a unique lift to a path $\Gamma$ in $\mathbb{C}$ with $\Gamma(0) = \mu$. If we define $F(z) = \Gamma(1)$, show that $F(z)$ does not depend on the path $\gamma$ chosen and that $F$ has the required properties.]

(4) If the map $f$ of Question 3 is a conformal equivalence, show that $F(z) = \lambda z + \mu$ for some $\lambda \in \mathbb{C}^\ast$. Hence deduce that two analytic tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda \Lambda_1$ for some $\lambda \in \mathbb{C}^\ast$.

(5) Show that complex tori $\mathbb{C}/\langle 1, \tau_1 \rangle$ and $\mathbb{C}/\langle 1, \tau_2 \rangle$ are analytically isomorphic if and only if $\tau_2 = \pm(a \tau_1 + b)/(c \tau_1 + d)$, for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

(6) Show that the component of the space of germs over $\mathbb{C}^\ast$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbb{C}$. Show that the component of the space of germs over $\mathbb{C}\setminus\{-1, 0, 1\}$ corresponding to the complete analytic function $(z^3 - z)^{1/2}$ is analytically isomorphic to the Riemann surface we constructed by gluing.

(7) Let $R$ denote the Riemann surface associated with the complete analytic function $\sqrt{1 - \sqrt{z}}$ over $\mathbb{C}^\ast$. Show that the projection covering map to $\mathbb{C}^\ast$ is surjective. Find analytic continuations along homotopic curves in $\mathbb{C}^\ast$, say from $1/2$ to $3/2$, which have the same initial germ at $1/2$ but different final germs at $3/2$. Why is this consistent with the Classical Monodromy theorem?

(8) Consider the analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the ramification points of $f$ and the corresponding ramification indices. What are the branch points?
(9) Suppose that \( f : R \to S \) is an analytic map of compact Riemann surfaces, and let \( B \subset S \) denote the set of branch points. Show that the map \( f : R \setminus f^{-1}(B) \to S \setminus B \) is a regular covering map. [Hint: Similar argument to that used in the Valency theorem.] Given a point \( P \in S \setminus B \) and a closed curve \( \gamma \) in \( S \setminus B \) with initial and final point \( P \), explain how this defines a permutation of the (finite) set \( f^{-1}(P) \). Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre \( f^{-1}(P) \). What group is obtained in Question 8?

(10) Let \( f(z) = p(z)/q(z) \) be a rational function on \( C \), where \( p, q \) are coprime polynomials. Show that \( f \) defines an analytic map \( f : C_{\infty} \to C_{\infty} \), whose degree \( d \) is the maximum of the degrees of \( p \) and \( q \). If \( f' \) denotes the derivative of the function \( f \), show that it defines an analytic map \( f' : C_{\infty} \to C_{\infty} \), whose degree satisfies \( d - 1 \leq \text{deg} f' \leq 2d \). Give examples to demonstrate that the bounds can be achieved.

(11) If \( f : R \to S \) is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy \( g(R) \geq g(S) \). Show that any non-constant analytic map between compact Riemann surfaces of the same genus \( g > 1 \) must be an analytic isomorphism. Does this last statement hold when \( g = 0 \) or 1?

(12) Let \( \pi : R \to C \setminus \{1, i, -1, -i\} \) be the Riemann surface associated to the complete analytic function \((z^4 - 1)^{1/4}\). Describe \( R \) explicitly by a gluing construction. Assuming the fact that \( R \) may be compactified to a compact Riemann surface \( \bar{R} \) and \( \pi \) extended to an analytic map \( \bar{\pi} : \bar{R} \to C_{\infty} \), find the genus of \( \bar{R} \).