1. If $f$ is a meromorphic doubly-periodic (i.e. elliptic) function of degree $k > 0$ show that $f'$ is an elliptic function whose degree $\ell$ satisfies $k + 1 \leq \ell \leq 2k$. Give examples to show that both bounds are attained.

Recall from example sheet 1: $\psi(z, \tau) = \sum_{n=-\infty}^{\infty} e^{(1/2)(n + 1/2)^2 \tau + (n + 1/2)(z + 1/2)}$ and satisfies $\psi(z + 1) = -\psi(z)$, $\psi(z + \tau) = -e^{(-1/2 - z)}\psi(z)$, where $e(z) = \exp(2\pi iz)$, $\psi(z) = -\psi(-z)$, and has unique zero ‘modulo the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$'.

2. (i) Prove that if $z, w \in \mathbb{C}$, then
\[
\varphi(z) - \varphi(w) = -\psi'(0)2\psi(z - w)\psi(z + w)/\psi(z)^2\psi(w)^2.
\]

[Hint: Regarding one of $w, z$ as parameter, prove that each side is $\Lambda$-periodic in the other variable and has same zeros and poles. Get multiplicative constant by considering Laurent expansion at zero.]

(ii) Deduce that $\psi'(z) = -\psi'(0)\psi(2z)/\psi(z)^3$ and recover from this formula the zeros of $\psi'$.

3.* Elliptic functions may be thought of as generalizations of trigonometric functions. To make this more precise, consider $\psi(z, it)$ for $t > 0$. Show that for each fixed $z$,
\[
\exp(\pi t/4)\psi(z, it) \rightarrow -2\sin(\pi z), \quad t \rightarrow \infty.
\]

This suggests the replacement
\[
\psi(z, it) = \psi_\infty(z) = -2\sin\pi z,
\]
\[
\chi(z, it) = \psi'(z, it)/\psi(z, it) \quad \text{by} \quad \chi_\infty(z) = \psi'_\infty(z)/\psi_\infty(z) = \pi\cot\pi z,
\]
\[
\varphi(z, it) = \text{const} - \chi'(z, it) \quad \text{(explain) by} \quad \varphi_\infty(z) = \text{const} - \chi'_\infty(z) = \text{const} + \pi^2/\sin^2\pi z.
\]

Verify that in order that $\varphi_\infty(z) = 1/z^2 + z^2 \cdot (\text{holomorphic function near zero})$,
we must have $\varphi_\infty(z) = \pi^2/\sin^2\pi z - \pi^2/3$.

Verify also that $\varphi_\infty$ satisfies the differential equation for $\varphi$ for suitable values of $E_4$ and $E_6$ (find these values!)

4. Denote by $e_1, e_2, e_3$ the values $\varphi(1/2), \varphi(\tau/2), \varphi((1 + \tau)/2)$ of $\varphi$ at the half-periods.

(i) Show that $e_1 + e_2 + e_3 = 0$. Obtain expressions for $e_1^3 + e_2^3 + e_3^3$ and $e_1^2 + e_2^2 + e_3^2$ in terms of the coefficients $g_2, g_3$ of the differential equation $(\varphi'(z))^2 = 4\varphi^3(z) - 2g_2\varphi(z) - g_3$.

(ii)* Show that $e_1, e_2, e_3$ are pair-wise distinct. [Hint: the zeros of $\varphi'$.

5. Prove the addition theorem for $\varphi$,
\[
\begin{vmatrix}
1 & 1 & 1 \\
\varphi(u) & \varphi(v) & \varphi(w) \\
\varphi'(u) & \varphi'(v) & \varphi'(w)
\end{vmatrix} = 0
\]
if and only if two of $u, v, w$ are congruent modulo $\Lambda$ or $u + v + w \in \Lambda$.

[Hint: consider the determinant as a function of $u$ with parameters $v, w$. The case $v + w \in \Lambda$ is exceptional (why?)].
6. Show that any holomorphic map \( f \) of degree 2 from an elliptic curve \( \mathbb{C}/\Lambda \) to \( S^2 \) is given by a ‘Möbius transformation of a shifted \( \wp \)-function’:
\[
 f(z) = \frac{a \wp(z - z_0) + b}{c \wp(z - z_0) + d},
\]
for some \( a, b, c, d, z_0 \in \mathbb{C} \).

7. Show, by considering the unit disc \( \Delta \) and the complex plane \( \mathbb{C} \), that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic).
Show that no two of the following domains in \( \mathbb{C} \) are conformally equivalent
\[
\{ 1 < |z| < 2 \}, \quad \{ 0 < |z| < 1 \}, \quad \{ 0 < |z| < \infty \}.
\]

8. (i) Let \( R \) and \( S \) be some Riemann surfaces, \( f : R \to S \) a continuous map, and \( p \) a point in \( R \). Show, directly from the definition of holomorphic maps, that if \( f \) is holomorphic on \( R \setminus \{ p \} \) then \( f \) is in fact holomorphic on all of \( R \).

(ii) Suppose that each of \( A = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) and \( B = \{ \beta_1, \beta_2, \beta_3, \beta_4 \} \) is a set of four distinct points in \( S^2 \) and \( F : S^2 \setminus A \to S^2 \setminus B \) is a biholomorphic map. Show that \( F \) extends to a biholomorphic map of \( S^2 \) onto itself, hence the \( \beta_i \) is constrained to be in a finite subset of \( S^2 \) determined by the other \( \beta_i \)’s and \( \alpha_j \)’s.

9. Show that if \( R \) and \( S \) are Riemann surfaces such that both are connected, \( R \) is compact and \( S \) is non-compact then every holomorphic map \( f : R \to S \) is constant.

10. (i) Let \( R \) and \( S \) be compact connected Riemann surfaces and \( g : R \to S \) a non-constant holomorphic map. Show that the genus of \( R \) is greater or equal to the genus of \( S \).

(ii) Let \( R \) and \( S \) be compact connected Riemann surfaces, such that
\[
\text{genus}(R) = \text{genus}(S) = g.
\]
Show that if \( f : R \to S \) is a non-constant holomorphic map and \( g > 1 \) then \( f \) is biholomorphic. What does the argument give in the case when (a) \( g = 0 \) or (b) \( g = 1 \)?

(iii) Show that a holomorphic map \( f : S^2 \to S^2 \) of degree \( k \geq 2 \) must have ramification points (i.e. points \( p \in S^2 \) with \( \vartheta_f(p) > 1 \)); recover from this the answer to Q7 in ex. sheet 1.

11. (i) Let \( f \) and \( g \) be two elliptic functions (with the same lattice of periods) and \( N \) a positive integer. By considering the poles of \( f \) and \( g \), estimate from above the dimension of the complex vector space spanned by \( f(z)^m g(z)^n \), for \( 0 \leq m, n \leq N \). Deduce that when \( N \) is sufficiently large there must be a non-trivial linear dependence,
\[
\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \quad \text{for some} \ a_{m,n} \in \mathbb{C}.
\]
Hence show that any two meromorphic functions \( f, g \) on an elliptic curve \( \mathbb{C}/\Lambda \) are ‘algebraically related’: there is a polynomial \( Q \) in two variables, so that \( Q(f(z), g(z)) = 0 \) for all \( z \).

(ii)* Show that in fact (i) holds for meromorphic functions on any compact Riemann surface.

12. Recall from the Lectures that \( \vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2 \tau + nz) \), where \( \mathbf{e}(z) = \exp(2\pi iz) \) and \( \text{Im}(\tau) > 0 \). Show that if \( k \) is a positive integer then \( \vartheta(0, \tau)^k = \sum_{n=0}^{\infty} r_n(k)e^{\pi i nt} \), where \( r_n(k) \) is the number of ways to express the integer \( n \) as a sum of \( k \) squares.

Supervisors can obtain an annotated version of this example sheet from DPMMS.