

Part IID RIEMANN SURFACES (2006–2007): Example Sheet 3

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1. Show that for any non-constant complex polynomial $P(s, t)$, the set $\{(s, t) \in \mathbb{C}^2 : P(s, t) = 0\}$ is unbounded. (I.e. any algebraic curve in \mathbb{C}^2 is non-compact.)

2. Let $S_0 = \{(s, t) \in \mathbb{C}^2 : t^2 = s^2 - a^2\}$, where a is a fixed non-zero complex number. Show that S_0 is a non-singular curve.

By finding the intersection point(s) of S_0 with the complex line $\lambda(s - a) = t$, show that the map $\varphi : \mathbb{C} \setminus \{1, -1\} \rightarrow S_0 \setminus \{(a, 0)\}$ given by

$$\varphi(\lambda) = \left(a \frac{\lambda^2 + 1}{\lambda^2 - 1}, \frac{2a\lambda}{\lambda^2 - 1} \right)$$

is biholomorphic. Thus φ can be thought of as a ‘parameterization’ of an open subset in S_0 .

3. Identify the non-singular projective curve S so that S_0 is biholomorphic to an open subset $\{X : Y : Z \in S \mid X \neq 0\}$ and write down the points of $S \setminus S_0$ (the points at ‘infinity’ of S).

Show that φ (as defined in Question 2) extends to a holomorphic map $\varphi : \mathbb{C} \rightarrow S$. Determine $\varphi(z)$, for $z = \pm 1$, and $\varphi(\mathbb{C})$.

Finally, identify \mathbb{C} as $\mathbb{P}^1 - \{1 : 0\}$ via $\lambda \mapsto \lambda : 1$ and show, by further extending φ , that S is biholomorphic to the Riemann sphere.

[Hint: verifying that φ extends continuously to \mathbb{P}^1 will suffice — explain why.]

4. (Projective transformations.) Show that any linear isomorphism $A \in GL(3, \mathbb{C})$ induces a homeomorphism (still to be denoted by A) of the projective plane \mathbb{P}^2 onto itself. When do $A, B \in GL(3, \mathbb{C})$ induce the same map on \mathbb{P}^2 ?

Show further that the restriction of A to any non-singular projective curve $C \subset \mathbb{P}^2$ gives a biholomorphic map of C onto the image $A(C)$.

5. Let $E = \mathbb{C}/\Lambda$ be the elliptic curve defined by a lattice Λ and write $E_0 = E \setminus \{\Lambda\}$ for the complement of the coset of Λ . Show that

$$\Phi : z + \Lambda \in E_0 \rightarrow (\wp(z), \wp'(z)) \in \mathbb{C}^2$$

maps the punctured elliptic curve E_0 biholomorphically onto a non-singular algebraic curve in \mathbb{C}^2 .

Show further that Φ extends to a biholomorphic map of E onto a non-singular projective curve $\{P(X, Y, Z) = 0\}$, for a certain homogeneous cubic polynomial P .

[Hint: the differential equation for \wp .]

6. (Hyperelliptic involution.) A compact Riemann surface S is called **hyperelliptic** if it admits a meromorphic function $f : S \rightarrow \mathbb{P}^1$ of degree 2. Show that, for any hyperelliptic Riemann surface S , the map $a : S \rightarrow S$ determined (uniquely) by the properties $f \circ a = f$, and $a(x) \neq x$ if $v_f(x) = 1$, is holomorphic.

7. Consider the complex algebraic curve C in \mathbb{C}^2 defined by the vanishing of the polynomial $P(s, t) = t^3 - s(s^2 - 1)$. Show that C is non-singular and find the branch locus of the branched cover $f : C \rightarrow \mathbb{C}$ given by the first projection. Find also the ramification points of f and the branching orders.

8. Analyze a compactification of the curve C of Question 7 along the following lines.

(i) For $|z| > 1$, show that there exists a holomorphic function $h(z)$ such that $h(z)^3 = 1 - z^{-2}$, $h(z) \rightarrow 1$ as $|z| \rightarrow \infty$.

(ii) Deduce, by writing the equation for C in the form $t^3 = (s \cdot h(s))^3$, that $C \cap \{(s, t) \in \mathbb{C}^2 : |s| > 1\} = C_1 \cup C_2 \cup C_3$, where the C_j are pairwise disjoint and the restriction of f to C_j gives a biholomorphic map to $\{s \in \mathbb{C} : |s| > 1\}$.

(iii) Hence show that there exists a compact Riemann surface $R = C \cup \{\infty_1\} \cup \{\infty_2\} \cup \{\infty_3\}$ together with a holomorphic map $F : R \rightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, such that the restriction of F to C is f and $F(\infty_j) = \infty$ ($j = 1, 2, 3$).

Now find the genus of the surface R .

9. Prove that two divisors on the Riemann sphere are linearly equivalent if and only if they have the same degree.

10. Let D be an effective divisor on \mathbb{P}^1 . Show, directly from the definition of ℓ , that $\ell(D) = \deg D + 1$.

11. The curves in this question are assumed to be connected. Suppose that C is a non-singular complex projective curve and P a point in C with $\ell(P) > 1$. If $f \in \mathcal{L}(P)$ is non-constant, show that the map $\alpha : x \in C \rightarrow f(x) : 1 \in \mathbb{P}^1$ is biholomorphic. ($\alpha(x) = 1 : 0$ if x is a pole of f .) Show further that if D is an effective non-zero divisor on a non-singular complex projective curve not biholomorphic to \mathbb{P}^1 then $\ell(D) \leq \deg D$.

12. Let φ and ψ denote the charts on the Riemann sphere S^2 defined by the stereographic projections from, respectively, the North and South poles.

(i) Regarding the differential dz on \mathbb{C} as a local expression for a differential on S^2 with respect to φ , determine the corresponding local expression for this differential with respect to ψ . Hence show that dz extends to a meromorphic differential on the Riemann sphere and has a pole at ‘infinity’. Find the order of this pole.

(ii) More generally, if $(z - a)^n dz$ ($a \in \mathbb{C}$, $n \in \mathbb{Z}$) is a formula for a meromorphic differential $\omega_{a,n}$ on S^2 relative to φ give the formula for this differential relative to ψ . Write down the divisor of $\omega_{a,n}$.

(iii) Show also that there are no non-zero holomorphic differentials on S^2 .

Does the holomorphic differential $e^z dz$ on \mathbb{C} extend to a meromorphic differential on the Riemann sphere? Justify your answer.

13. Show that the differential dz on \mathbb{C} induces a well-defined **holomorphic** differential η on an elliptic curve \mathbb{C}/Λ , via the standard charts given by local inverses of the quotient map $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$. Deduce that the differential η is **not** obtainable as df for any meromorphic function f on \mathbb{C}/Λ .

Find a pair of meromorphic functions f and g on \mathbb{C}/Λ , so that $\eta = fdg$.

14.* Verify the Riemann–Roch theorem for the special cases of the Riemann sphere and an elliptic curve. You may assume results of any of the above questions.