Part IID RIEMANN SURFACES (2004–2005): Revision Example Sheet

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Note. The present 24-lecture course on Riemann Surfaces was developed from the 16-lecture IIB course with the same title by incorporating some topics from IIB Algebraic Curves. Past exam questions on IIB Riemann surfaces are mostly suitable for revision. In fact, many of the questions on this sheet are borrowed (possibly with modifications) from past exam papers. Most of the past exam questions on Algebraic Curves are not quite compatible with the present course (not before some editing, anyway).

It is never a bad idea to use for revision some of those questions from Sheets 1–4 (particularly non-* ones) for which you did not have enough time during the term.

1. Show that any holomorphic map from the Riemann sphere to itself corresponds, via stereographic projection, to a rational function on $\mathbb{C}$. Find the most general form of a biholomorphic map of the Riemann sphere onto itself.

Deduce from the Weierstrass–Casorati Theorem that a biholomorphic map f of $\mathbb{C}$ onto itself cannot have an essential singularity at infinity. Hence determine the most general form of such f.

2. Let P and Q be complex polynomials of degree $m \geq 2$ with no common roots. Explain briefly how the rational function $P(z)/Q(z)$ induces a holomorphic map F from the 2-sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ to itself. What is the degree of F?

Let f and g be two holomorphic maps on the Riemann sphere $\mathbb{C} \cup \infty$. Show that

$$|\deg f - \deg g| \leq \deg(f+g) \leq \deg f + \deg g.$$

3. Let Λ be a lattice in $\mathbb{C}$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, where $\operatorname{Im}\tau > 0$. Describe briefly a construction of a family of charts making an elliptic curve $\mathbb{C}/\Lambda$ into a Riemann surface, so that the quotient map $\pi : z \in \mathbb{C} \to z + \Lambda \in \mathbb{C}/\Lambda$ is a holomorphic map.

Let f be an elliptic function with respect to Λ and suppose that f has degree d. Let A be the set of points $c \in \mathbb{C} \cup \{\infty\}$ for which the equation $f(u) = c$ has *less* than d distinct solutions (cosets) $u + \Lambda$. Prove that A is a non-empty finite set.

4. Let $h(z)$ be a Λ-periodic meromorphic function, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ is a lattice in $\mathbb{C}$, $\operatorname{Im}\tau > 0$. Suppose that $h(z)$ has at the points $\frac{1}{2}(m + n\tau)$, $m, n \in \mathbb{Z}$, simple zeros precisely when $m + n$ is odd, and simple poles precisely when $m + n$ is even, and no other zeros or poles. Show that $h(z)$ is a constant multiple of $\dfrac{\wp'(z)}{\wp(z) - \wp(\frac{1+\tau}{2})}$ Here $\wp$ denotes the Weierstrass π-function with respect to the lattice Λ.

[Standard properties of the doubly-periodic meromorphic functions may be assumed without proof if accurately stated, but is you use properties of the $\wp$-function these should be deduced from first principles.]

5. Let $f : R \to S$ be a non-constant holomorphic map between compact connected Riemann surfaces. Define the *branching order* of f at a point, and the *degree* of f. Define the *genus* of a compact connected Riemann surface (assuming the existence of a triangulation). State the Riemann–Hurwitz theorem.

Describe briefly how to define a family of charts making an non-singular algebraic curve in $\mathbb{P}^2$ into a Riemann surface (Implicit Function Theorem should be assumed without proof.) Fermat curve is a projective curve defined as $F_d = \{X : Y : Z \in \mathbb{P}^2 \mid X^d + Y^d = Z^d\}$, for an integer $d \geq 2$. Verify that Fermat curve is non-singular. By considering a suitable holomorphic map of $F_d \to \mathbb{P}^1$, or otherwise, find the genus of F_d.

6. Let f, g be two meromorphic functions on a Riemann surface. Explain what is meant by a meromorphic differential fdg, and by a zero and a pole of fdg.

Let C_0 is a curve in $\mathbb{C}^2$ defined by the vanishing of polynomial $p(s,t) = t^2 - t - s^3$. Write out the equation of the corresponding projective curve in $C \subset \mathbb{P}^2$ and show that C is non-singular. Let $f_1 : (s,t) \in C_0 \to s \in \mathbb{C}$s and $f_2 : (s,t) \in C_0 \to t \in \mathbb{C}$ denote the restrictions to C_0 of the first and the second projection. Show that $\omega = \dfrac{df_1}{2f_2 - 1}$ defines a holomorphic differential on C_0. Does ω have zeros on C_0?

7. In this question S is a compact connected Riemann surface and you may assume that S admits non-constant meromorphic functions.

Define *canonical divisors* on a compact connected Riemann surface S and show that any two canonical divisors are linearly equivalent. State the Riemann–Roch theorem and deduce from it that the degree of a canonical divisor on S is determined by the topological surface underlying S.

Show that if S has genus 0 then S is biholomorphic to the Riemann sphere.

8. Let $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$. Suppose that f is a power series that converges in some neighbourhood of the point 1 in $\mathbb{C}^*$, and that f can be analytically continued over every curve in $\mathbb{C}^*$ that starts at the point 1. Let $f_1(z)$ be the power series that converges in some neighbourhood of the point 1, and that is obtained by the analytic continuation of f around the circle $e^{2\pi i t}$, $0 \leq t \leq 1$. Show that if $f_1(z) = f(z)$ then f is single-valued on $\mathbb{C}^*$.

Describe how f determines a simply-periodic function F in the complex plane $\mathbb{C}$. What is F when $f(z) = z^k$, where k is a positive integer?

[Any form of the Monodromy Theorem may be used without proof provided that it is carefully stated. Elementary properties of homotopy may be used without proof. You might like to consider the universal covering $z \mapsto \exp(z)$ of $\mathbb{C}^*$.]