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1. If \( f \) is a meromorphic doubly-periodic (i.e. elliptic) function of degree \( k > 0 \) show that \( f' \) is an elliptic function whose degree \( \ell \) satisfies \( k + 1 \leq \ell \leq 2k \). Give examples to show that both bounds are attained.

Recall from example sheet 1: \( \psi(z, \tau) = \sum_{n=1}^{\infty} e^{\frac{1}{2}(n + \frac{1}{2})^2 \tau + (n + \frac{1}{2})(z + \frac{1}{2})} \) and satisfies \( \psi(z + 1) = -\psi(z), \ \psi(z + \tau) = -e^{-\frac{\tau}{2} - z} \psi(z) \), where \( e(z) = \exp(2\pi iz) \), \( \psi(z) = -\psi(-z) \), and has unique zero ‘modulo the lattice \( \mathbb{Z} + \tau \mathbb{Z} \).

2. (i) Prove that if \( Z \) and has unique zero ‘modulo the lattice \( \mathbb{Z} + \tau \mathbb{Z} \) is an elliptic function whose degree \( \ell \) satisfies \( k + 1 \leq \ell \leq 2k \). Find these values!

(ii) Deduce that \( \psi(z) = -\psi(0)^3 \psi(2z) \psi(z) \psi(z) \) and recover from this formula the zeros of \( \psi' \).

3. Let \( \chi(z) = \psi'(z)/\psi(z) \). Differentiate 2(i) and interchange \( z \) and \( w \) to obtain:

\[
\frac{1}{2} \left( \frac{\psi'(z) - \psi'(w)}{\psi(z) - \psi(w)} \right) = \chi(z + w) - \chi(z) - \chi(w).
\]

Remark for readers of Ahlfors or Jones & Singerman: their \( \sigma \) and \( \zeta \) are not quite the same as \( \psi \) and \( \chi \) here, but for some constants \( A, B, \sigma(z) = \exp(Az^2 + B)\psi(z) \), so \( \zeta(z) = 2Az + \chi(z) \).

4.* (challenging but feasible) Prove the addition formula for \( \psi \),

\[
\psi(z + w) = -\psi(z) - \psi(w) + \frac{1}{4} \left( \frac{\psi'(z) - \psi'(w)}{\psi(z) - \psi(w)} \right)^2.
\]

You will need to differentiate the formula in Q3 and use the differential equation satisfied by \( \psi \) to eliminate \( \psi'' \). Note also that \( \psi = a - \chi' \), for some constant \( a \in \mathbb{C} \) (can you see why?).

5.* Elliptic functions may be thought of as generalizations of trigonometric functions. To make this more precise, consider \( \psi(z, it) \) for \( t > 0 \). Show that for each fixed \( z \),

\[\exp(\pi t/4)\psi(z, it) \rightarrow -2\sin(\pi z), \text{ as } t \rightarrow \infty.\]

This suggests the replacement

\[
\psi(z, it) \text{ by } \psi_\infty(z) = -2\sin \pi z, \\
\chi(z, it) \text{ by } \chi_\infty(z) = \psi_\infty'(z)/\psi_\infty(z) = \pi \cot \pi z, \\
\varphi(z, it) \text{ by } \varphi_\infty(z) = \text{const} - \chi_\infty'(z) = \text{const} + \pi^2 / \sin^2 \pi z.
\]

Verify that in order that \( \psi_\infty(z) = 1/\pi^2 + z^2 \) \cdot (holomorphic function near zero),

we must have \( \varphi_\infty(z) = \pi^2 \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} \).

Verify also that \( \varphi_\infty \) satisfies the differential equation for \( \varphi \) for suitable values of \( E_4 \) and \( E_6 \) (find these values!).
6. Show that any holomorphic map \( f \) of degree 2 from an elliptic curve \( \mathbb{C}/\Lambda \) to \( S^2 \) is given by a ‘Möbius transformation of a shifted \( \wp \)-function’:
\[
f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},
\]
for some \( a, b, c, d, z_0 \in \mathbb{C} \).

7. Show, by considering the unit disc \( \Delta \) and the complex plane \( \mathbb{C} \), that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic). Show that no two of the following domains are conformally equivalent
\[
\{ z : 1 < |z| < 2 \}, \quad \{ z : 0 < |z| < 1 \}, \quad \{ z : 0 < |z| < \infty \}.
\]

8. (i) Let \( R \) and \( S \) be some Riemann surfaces, \( f : R \to S \) a continuous map, and \( p \) a point in \( R \). Show, directly from the definition of holomorphic maps, that if \( f \) is holomorphic on \( R \setminus \{p\} \) then \( f \) is in fact holomorphic on all of \( R \).

(ii) Suppose that each of \( A = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) and \( B = \{ \beta_1, \beta_2, \beta_3, \beta_4 \} \) is a set of four distinct points in \( S^2 \) and \( F : S^2 \setminus A \to S^2 \setminus B \) is a biholomorphic map. Show that \( F \) extends to a biholomorphic map of \( S^2 \) onto itself, hence the \( \beta_i \)'s are determined by the other \( \beta_i \)'s and \( \alpha_j \)'s.

9. Show that if \( R \) and \( S \) are Riemann surfaces such that both are connected, \( R \) is compact and \( S \) is non-compact then every holomorphic map \( f : R \to S \) is constant.

10. (i) Let \( R \) and \( S \) be compact connected Riemann surfaces and \( g : R \to S \) a non-constant holomorphic map. Show that the genus of \( R \) is greater or equal to the genus of \( S \).

(ii) Let \( R \) and \( S \) be compact connected Riemann surfaces, such that
\[
\text{genus}(R) = \text{genus}(S) = g.
\]
Show that if \( f : R \to S \) is a non-constant holomorphic map and \( g > 1 \) then \( f \) is biholomorphic. What does the argument give in the case when (a) \( g = 0 \) or (b) \( g = 1 \)?

(iii) Show that a holomorphic map \( f : S^2 \to S^2 \) of degree \( k \geq 2 \) must have ramification points (i.e. points \( p \in S^2 \) with \( v_f(p) > 1 \)); recover from this fact the answer to Q7 in example sheet 1.

11. (i) Let \( f \) and \( g \) be two elliptic functions (same lattice of periods) and \( N \) a positive integer. By considering the poles of \( f \) and \( g \), estimate from above the dimension of the complex vector space spanned by \( f(z)^m g(z)^n \), for \( 0 \leq m, n \leq N \). Deduce that when \( N \) is sufficiently large there must be a non-trivial linear dependence,
\[
\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \quad \text{for some } a_{m,n} \in \mathbb{C}.
\]
Hence show that any two meromorphic functions \( f, g \) on an elliptic curve \( \mathbb{C}/\Lambda \) are ‘algebraically related’: there is a polynomial \( Q \) in two variables, so that \( Q(f(z), g(z)) = 0 \) for all \( z \).

(ii)* Show that the argument of (i) works for meromorphic functions on any compact Riemann surface.

12. Recall from the Lectures that \( \vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\frac{\pi}{\tau}(n^2 \tau + nz)} \), where \( e(z) = \exp(2\pi iz) \) and \( \text{Im}(\tau) > 0 \). Show that if \( k \) is a positive integer then \( \vartheta(0, \tau)^k = \sum_{n=0}^{\infty} r_n(k) e^{\pi i n \tau} \), where \( r_n(k) \) is the number of ways to express the integer \( n \) as a sum of \( k \) squares.