Representation Theory — Examples Sheet 1

1. Let ρ be a representation of a group G. Show that $\det \rho$ is a representation of G. What is its degree?

2. Let θ be a one-dimensional representation of a group G and $\rho\colon G \to GL(V)$ another representation of G. Show that the map $\theta \otimes \rho\colon G \to GL(V)$ given by $\theta \otimes \rho(g) = \theta(g) \cdot \rho(g)$ defines a representation of G. If ρ is irreducible, must $\theta \otimes \rho$ also be irreducible?

3. Suppose that N is a normal subgroup of a group G. Given a representation of the quotient group G/N on a vector space V, explain how to construct an associated representation of G on V. Which representations of G arise in the way? Recall that G' is the normal subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$ with $g, h \in G$. Show that the 1-dimensional representations of G are precisely those that arise from 1-dimensional representations of G/G'.

4. Suppose that (ρ, V) and (σ, W) are representations of a group G. Show that $(\tau, \mathrm{Hom}_k(V, W))$ is a representation of G where $\tau(g)(\alpha) := \sigma(g) \circ \alpha \circ \rho(g^{-1})$ for all $g \in G$ and $\alpha \in \mathrm{Hom}_k(V, W)$. Show that $\mathrm{Hom}_G(V, W)$ is a subrepresentation, on which the elements of G act trivially.

5. Let C_n be the cyclic group of order n. Explicitly decompose the complex regular representation $\mathbb{C}C_n$ as a direct sum of irreducible subrepresentations.

6. Let $\rho\colon \mathbb{Z} \to GL_2(\mathbb{C})$ be the representation defined by $\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that ρ is not completely reducible. By a similar construction, show that if k is a field of characteristic p there is a two dimensional k-representation of C_p that is not completely reducible. *Optional: If k algebraically closed and of characteristic p, for which n is every k-representation of the cyclic group C_n completely reducible?*

7. Let D_{10} be the dihedral group of order 10. This is generated by elements r, t, with $r^5 = 1 = t^2$, $trt = r^{-1}$.. Let V be an irreducible complex representation of D_{10}. Pick an eigenvector v for r, and show the subspace spanned by v, tv is preserved by the action of D_{10}. Hence conclude that every irreducible complex representation of D_{10} has degree 1 or 2. By describing them explicitly, show that there are precisely four such representations up to isomorphism. Show moreover that for each such representation it is possible to choose a basis so that all the representing matrices have real entries.

8. Show that (up to isomorphism) there is only one irreducible complex representation of Q_8 of dimension at least two. Show that this representation cannot be realised over $\mathbb{R}$ and deduce that that Q_8 is not isomorphic to a subgroup of $GL_2(\mathbb{R})$. Find a four-dimensional irreducible real representation V of Q_8. Compute $\mathrm{Hom}_G(V, V)$ in this case.

9. What are the irreducible real representations $\rho\colon C_n \to GL(V)$ of a cyclic group of order n? Compute $\mathrm{Hom}_G(V, V)$ in each case. How does the real regular representation $\mathbb{R}C_n$ of C_n break up as a direct sum of irreducible representations?

10. We showed in class that Jordan normal form implies that a representation of a finite *cyclic* group on a complex vector space V is a direct sum of one dimensional representations. Using this, show that this is still true for finite *abelian* groups G. [Hint: Pick $g \in G$. Show that the decomposition of V into eigenspaces of g is preserved by all $h \in G$. Now pick $h \in G \setminus \{g\}$, and decompose each eigenspace of g into eigenspaces of h. Repeat.] In particular, all irreducible representations are one dimensional.

11. Using the description of an abelian group as a product of cyclic groups, describe each one dimensional complex representation of a finite abelian group G. Show that there are exactly $|G|$ such. *Optional: When does such a G have a faithful irreducible representation?*

12. Let G act on a complex representation V, and let $v \in V$. Show that averaging over the orbit of G on v defines a vector $\overline{v}$ fixed by G. Now suppose V is an irreducible representation of G, and V is not the trivial representation. What can you say about $\overline{v}$?

13. *Optional question.* Let G act on a complex representation V, and let $\langle , \rangle : V \times V \to \mathbf{C}$ be a skew-symmetric (= alternating) form, that is $\langle x, y \rangle = -\langle y, x \rangle$ for all $x, y \in V$. Show that $(x, y) := 1/|G| \sum_{g \in G} \langle gx, gy \rangle$ is a G-invariant skew-symmetric form.

    Does this imply every finite subgroup of $GL_{2n}(\mathbf{C})$ is conjugate to a subgroup of $Sp_{2n}(\mathbf{C}) = \{g \in GL_{2n} \mid \langle gx, gy \rangle = \langle x, y \rangle \text{ for all } x, y\}$?

14. Show that if $\rho\colon G \to GL(V)$ is a representation of a finite group G on a real vector space V then there is a basis for V with repect to which the matrix representing $\rho(g)$ is orthogonal for every $g \in G$. Which finite groups have a faithful two-dimensional real representation?

15. Let (ρ, V) be an irreducible complex representation of a finite group G. For each $v \in V$, show that the $\mathbb{C}$-linear map $\mathbb{C}G \to V$ given by $e_g \mapsto \rho(g)(v)$ is G-linear and deduce that V is isomorphic to a quotient representation of $\mathbb{C}G$. Conclude V is a summand of $\mathbb{C}G$. What is $\dim \mathrm{Hom}_G(\mathbb{C}G, V)$?

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