Representation Theory — Examples Sheet 4

On this sheet all representations are complex representations unless stated otherwise.

1. Let $SU(2)$ act on the space $M_3(\mathbb{C})$ of 3×3 complex matrices by
$$A\colon X\mapsto A_1XA_1^{-1},$$
where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A,1$. Show that this defines a representation of $SU(2)$ and decompose it into irreducibles.

2. Let χ_n be the character of the irreducible representation of $SU(2)$ of dimension $n+1$. Show that
$$\frac{1}{2\pi}\int_0^{2\pi}K(z)\overline{\chi_n}\chi_m\,\mathrm{d}\theta=\delta_{nm},$$
where $z=e^{i\theta}$ and $K(z)=-\frac{1}{2}(z-z^{-1})^2$.

3. Let $G=SU(2)$ and V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y viewed as an irreducible representation of G.
   (a) Show that V_n is isomorphic to its dual V_n^* as representations of G.
   (b) Decompose the representations $V_4\otimes V_3$, $V_3\otimes V_3$, S^2V_3 and Λ^2V_3 into irreducibles.
   (c) How do $V_1^{\otimes n}$, S^nV_1, S^2V_n and Λ^2V_n decompose into irreducibles for $n\geq 1$. What about S^3V_2?

4. By considering the action of $SU(2)$ by conjugation on the vector space of 2×2 complex matrices A such that $A=-\overline{A}^T$ and $\operatorname{tr}A=0$, equipped with norm $||A||^2=\det A$, construct a continuous group homomorphism $SU(2)\to SO(3)$. Deduce that $SU(2)/\{\pm I\}\cong SO(3)$ as topological groups.

5. Write down a Haar integral on $SU(2)$ and prove that it is translation invariant and normalised correctly.

6. Let G be a compact group. Show that if G has an n-dimensional faithful representation over $\mathbb{R}$ then there is a continuous faithful group homomorphism from G to the orthogonal group $O(n)$.

7. Let $G=PSL_2(\mathbb{F}_7)=SL_2(\mathbb{F}_7)/Z(SL_2(\mathbb{F}_7))$. Starting with the character table of $GL_2(\mathbb{F}_7)$, calculate the character table of G. Deduce that G is simple. By considering the structure constants of $Z(\mathbb{C}G)$, and only using information in the character table, show that G has elements of order 2 and 3 whose product has order 7. Deduce that G is generated by two of its elements.

8. Let G be the topological group of 3×3 upper unitriangular real matrices
$$G:=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$$
Show that every representation of G of degree 1 factors through $G/Z(G)$.
Let
$$Z_0:=\left\{\begin{pmatrix}1&0&z\\0&1&0\\0&0&1\end{pmatrix}:z\in\mathbb{Z}\right\}\leq G$$
By considering restriction from G/Z_0 to $Z(G)/Z_0\cong S^1$ and determinants show that G/Z_0 has no faithful representations.

*9. Let $\mathbb{F}$ be the field with 2^n elements for some $n\geq 1$. Construct the character table of $GL_2(\mathbb{F})$. Deduce that $PGL_2(\mathbb{F})=GL_2(\mathbb{F})/Z(GL_2(\mathbb{F}))$ is simple for $n\geq 2$. What can you say about $PGL_2(\mathbb{F})$ when $n=1$?

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