

**MATHEMATICAL TRIPOS, PART II, 2020/2021  
REPRESENTATION THEORY EXAMPLE SHEET 4**

Unless otherwise stated, all vector spaces are finite-dimensional over  $\mathbb{C}$ . In the first seven questions we let  $G = \text{SU}(2)$ . Questions 9–12 deal with a variety of topics at Tripos standard.

**1** Let  $V_n$  be the vector space of complex homogeneous polynomials of degree  $n$  in the variables  $x$  and  $y$ . Describe a representation  $\rho_n$  of  $G$  on  $V_n$  and show that it is irreducible. What is its character? Show that  $V_n$  is isomorphic to its dual  $V_n^*$ .

**2** Using the properties of exterior and symmetric powers, together with the Clebsch-Gordan formula, decompose the following spaces into irreducible  $G$ -spaces (that is, find a direct sum of irreducible representations which is isomorphic to the given  $G$ -space; you are not being asked to find such an isomorphism explicitly).

- (i)  $V_4 \otimes V_3, V_3^{\otimes 2}, \Lambda^2 V_3$ ;
- (ii)  $V_1^{\otimes n}$ ;
- (iii)  $S^2 V_n, \Lambda^2 V_n$  ( $n \geq 1$ ),  $S^3 V_2$
- (iv)  $S^n V_1$  for  $n \geq 1$ .

**3** Let  $G$  act on the space  $M_3(\mathbb{C})$  of  $3 \times 3$  complex matrices, by conjugation:

$$A : X \mapsto A_1 X A_1^{-1},$$

where  $A_1$  is the  $3 \times 3$  block diagonal matrix with block diagonal entries  $A, 1, 1$ . Show that this gives a representation of  $G$  and decompose it into irreducible summands.

**4** Let  $\chi_n$  be the character of the irreducible representation  $\rho_n$  of  $G$  on  $V_n$  of dimension  $n + 1$ .

Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where  $z = e^{i\theta}$  and  $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$ .

[ Note that all you need to know about integrating on the circle is orthogonality of characters:  $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$ . This is really a question about Laurent polynomials. ]

**5** Check that the usual formula for integrating functions defined on  $S^3 \subseteq \mathbf{R}^4$  defines a  $G$ -invariant inner product on the vector space of integrable functions on

$$G = \text{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

**6** Either of the following ways can be used to identify  $\mathrm{SO}(3)$  with real projective 3-space  $\mathbb{R}P^3$  (the topological space of lines passing through the origin in  $\mathbb{R}^{n+1}$ ), and hence show that  $G/\{\pm I_2\} \cong \mathrm{SO}(3)$ .

(a) [Sketched in lectures: fill out the remaining details.] Let  $\mathbb{H}^0 = \{ai + bj + ck : a, b, c \in \mathbb{R}\}$  be the 3-dimensional space of *pure quaternions*, and let the quaternions of unit length,  $Q = \{q : \|q\| = 1\}$ , act on  $\mathbb{H}^0$  by conjugation  $h \mapsto qhq^{-1}$ . Show that this defines a rotation of  $S^2 \subseteq \mathbb{H}^0$ , so that  $G/\{\pm I_2\} = Q/\{\pm I_2\} \cong \mathrm{SO}(3)$ .

(b) [Needs some topological knowledge.]\* First project  $S^2$  onto its equatorial plane by  $(x, y, z) \mapsto \zeta = \frac{x+iy}{1-z}$ . Show that a rotation of  $S^2$  corresponds to a transformation of the form  $\zeta \mapsto \frac{a\zeta+b}{-b\zeta+\bar{a}}$ . Note that with  $a\bar{a} + b\bar{b} = 1$  we obtain an element of  $G$  and that  $(a, b)$  and  $(a', b')$  determine the same transformation if and only if  $(a', b') = (-a, -b)$ . Now replace  $G \cong S^3$  by the quotient space  $\mathbb{R}P^3$ .

**7** Compute the character of the representation  $S^n V_2$  of  $G$  for any  $n \geq 0$ . Calculate  $\dim_{\mathbb{C}}(S^n V_2)^G$  (by which we mean the subspace of  $S^n V_2$  where  $G$  acts trivially).

Deduce that the ring of complex polynomials in three variables  $x, y, z$  which are invariant under the action of  $\mathrm{SO}(3)$  is a polynomial ring. Find a generator for this polynomial ring.

**8** It is known that any finite subgroup of  $\mathrm{SO}(3)$  is isomorphic to precisely one of the following groups:

- the cyclic group  $\mathbb{Z}/n\mathbb{Z}$ ,  $n \geq 1$ , generated by a rotation by  $2\pi/n$  around an axis;
- the dihedral group  $D_{2m}$  of order  $2m$ ,  $m \geq 2$  (the group of rotational symmetries in 3-space of a plane containing a regular  $m$ -gon);
- $A_4$ , the group of rotations of a regular tetrahedron;
- $S_4$ , the group of rotations of a cube (or regular octahedron);
- $A_5$ , the group of rotations of a regular dodecahedron (or regular icosahedron).

Derive this classification (Hint: let  $G$  be a finite subgroup of  $\mathrm{SO}(3)$  and consider the action of  $G$  on the unit sphere.) By considering the homomorphism  $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ , classify the finite subgroups of  $\mathrm{SU}(2)$ .

9 The Heisenberg group of order  $p^3$  is the (non-abelian) subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of matrices over the finite field  $\mathbb{F}_p$  ( $p$  prime). Let  $H$  be the subgroup of  $G$  comprising matrices with  $a = 0$  and  $Z$  be the subgroup of  $G$  of matrices with  $a = b = 0$ .

(a) Show that  $Z = Z(G)$ , the centre of  $G$ , and that  $G/Z = \mathbb{F}_p^2$ . Note that this implies that the derived subgroup  $G'$  is contained in  $Z$ . [You can check by explicit computation that it equals  $Z$ , or you can deduce this from the list of irreducible representations found in (d) below.]

(b) Find all 1-dimensional representations of  $G$ .

(c) Let  $\psi : \mathbb{F}_p \rightarrow \mathbb{C}^\times$  be a non-trivial 1-dimensional representation of the cyclic group  $\mathbb{F}_p = \mathbb{Z}/p$ , and define a 1-dimensional representation  $\rho_\psi$  of  $H$  by

$$\rho_\psi \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that  $\text{Ind}_H^G \rho_\psi$  is an irreducible representation of  $G$ .

(d) Prove that the collection of representations constructed in (b) and (c) gives a complete list of all irreducible representations.

(e) Determine the character of the irreducible representation  $\text{Ind}_H^G \rho_\psi$ .

10 Recall the character table of  $G = \text{PSL}_2(7)$  from Sheet 2, q.9. Identify the columns corresponding to the elements  $x$  and  $y$  where  $x$  is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and  $y$  is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group  $G$  acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space  $(\mathbb{F}_7)^2$ ). Obtain the permutation character of this action and decompose it into irreducible characters.

\*(Harder) Show that the group  $G$  is generated by an element of order 2 and an element of order 3 whose product has order 7.

[Hint: for the last part use the formula that the number of pairs of elements conjugate to  $x$  and  $y$  respectively, whose product is conjugate to  $t$ , equals  $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$ , where the sum runs over all the irreducible characters of  $G$ , and  $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$ .]

11 Let  $J_{\lambda,n}$  be the  $n \times n$  Jordan block with eigenvalue  $\lambda \in K$  ( $K$  is any field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

(a) Compute  $J_{\lambda,n}^r$  for each  $r \geq 0$ .

(b) Let  $G$  be cyclic of order  $N$ , and let  $K$  be an algebraically closed field of characteristic  $p > 0$ . Determine all the representations of  $G$  on vector spaces over  $K$ , up to equivalence. Which are irreducible? Which are indecomposable?

Remark: Over  $\mathbb{C}$  irreducibility and indecomposability coincide but this can fail for modular representations.

**12** [For enthusiasts only. Part (a) requires knowledge of Galois Theory.]\*

(a) Let  $G$  be a cyclic group and let  $\chi$  be a (possibly reducible) character of  $G$ . Let  $S = \{g \in G : G = \langle g \rangle\}$  and assume that  $\chi(s) \neq 0$  for all  $s \in S$ . Show that

$$\sum_{s \in S} |\chi(s)|^2 \geq |S|.$$

(b) Deduce a theorem of Burnside: namely, let  $\chi$  be an irreducible character of  $G$  with  $\chi(1) > 1$ . Show that  $\chi(g) = 0$  for some  $g \in G$ . [Hint: partition  $G$  into equivalence classes by calling two elements of  $G$  equivalent if they generate the same cyclic subgroup of  $G$ .]

SM, Michaelmas Term 2020

Comments on and corrections to this sheet may be emailed to [sm@dpmms.cam.ac.uk](mailto:sm@dpmms.cam.ac.uk)