MATHEMATICAL TRIPOS, PART II, 2020/2021
REPRESENTATION THEORY EXAMPLE SHEET 3

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Recall the character table of S_4 from Sheet 2. Find all the characters of S_5 induced from the irreducible characters of S_4. Hence find the complete character table of S_5.

Repeat, replacing S_4 by the subgroup $\langle(12345), (2354)\rangle$ of order 20 in S_5.

2 Recall the construction of the character table of the dihedral group D_{10} of order 10 from Sheet 2.

(a) Use induction from the subgroup D_{10} of A_5 to A_5 to obtain the character table of A_5.

(b) Let G be the subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ consisting of upper triangular matrices. Compute the character table of G.

Hint: bear in mind that there is an isomorphism $G/Z \to D_{10}$.

3 Let H be a subgroup of the group G. Show that for every irreducible representation ρ for G there is an irreducible representation ρ' for H with ρ a component of the induced representation $\mathrm{Ind}_H^G\,\rho'$.

Prove that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G : A|$.

4 Obtain the character table of the dihedral group D_{2m} of order $2m$, by using induction from the cyclic subgroup C_m. [Hint: consider the cases m odd and m even separately, as for m even there are two conjugacy classes of reflections, whereas for m odd there is only one.]

5 Prove the transitivity of induction: if $H < K < G$ then

$$\mathrm{Ind}_K^G\,\mathrm{Ind}_H^K\,\rho \cong \mathrm{Ind}_H^G\rho$$

for any representation ρ of H.

6 (a) Let $V = U \oplus W$ be a direct sum of $\mathbb{C}G$-modules. Prove that both the symmetric square and the exterior square of V have submodules isomorphic to $U \otimes W$.

(b) Calculate $\chi_{\Lambda^2\rho}$ and $\chi_{S^2\rho}$, where ρ is the irreducible representation of dimension 2 of D_8; repeat this for Q_8. Which of these characters contains the trivial character in the two cases?

7 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d.

(a) Compute the dimension of S^nV and Λ^nV for all n.

(b) Let $g \in G$ and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of g on V. What are the eigenvalues of g on S^nV and Λ^nV?

(c) Let $f(t) = \det(g - tI)$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of f and χ_{Λ^nV}?

(d) Find a relationship between χ_{S^nV} and f.

8 Let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$. For any integer $r \leqslant \frac{n}{2}$, write X_r for the set of all r-element subsets of X, and let π_r be the permutation character of the action of G on X_r. Observe $\pi_r(1) = |X_r| = \binom{n}{r}$. If $0 \leqslant \ell \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_\ell \rangle = \ell + 1.$$

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G$, $\chi^{(n-1,1)}, \chi^{(n-2,2)}, \ldots, \chi^{(n-m,m)}$ such that for all $r \leqslant m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \cdots + \chi^{(n-r,r)}.$$

In particular the class functions $\pi_r - \pi_{r-1}$ are irreducible characters of S_n for $1 \leqslant r \leqslant n/2$ and equal to $\chi^{(n-r,r)}$.

9 Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation for G affording the character χ. Give the characters of the representations $V \otimes V$, S^2V and $\Lambda^2 V$ in terms of χ.

(i) Let W be another finite-dimensional representation with character ψ. Show that

$$\dim W^G = \frac{1}{|G|} \sum_{g \in G} \psi(g)$$

where $W^G = \{w \in W : gw = w \text{ for all } g \in G\}$.

(ii) Prove that if V is irreducible, $V \otimes V$ contains the trivial representation at most once.

(iii) Given any irreducible character χ of G, the *indicator* $\iota\chi$ of χ is defined by

$$\iota\chi = \frac{1}{|G|} \sum_{x \in G} \chi(x^2).$$

By using the decomposition $V \otimes V = S^2V \oplus \Lambda^2 V$, deduce that

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

Deduce that if $|G|$ is odd then G has only one real-valued irreducible character.
[Remark. The sign $+$, resp. $-$, indicates whether $\rho(G)$ preserves an orthogonal, respectively symplectic form on V, and whether or not the representation can be realised over the reals. You can read about it in Ch. 23 of James and Liebeck.]

10 Suppose that G is a Frobenius group with Frobenius kernel K. Show that

(i) $C_G(k) \leqslant K$ for all $1 \neq k \in K$.

(ii) if χ is a non-trivial irreducible character of K then $\mathrm{Ind}_K^G \chi$ is also irreducible with K not lying in its kernel. Hence explain how to construct the character table of G, given the character tables of K and G/K.
[Hints for (ii):

(a) First, show each element of $G \setminus K$ permutes the conjugacy classes in K, and fixes only the identity.

(b) Deduce that each element of $G \setminus K$ fixes only the trivial character of K.

(c) Use the Orbit-Stabilizer theorem to deduce that if χ is a non-trivial irreducible character of K then the number of distinct conjugates of χ is $|G : K|$.

(d) Use Frobenius reciprocity to show that if χ is as above and ϕ is an irreducible constituent of $\mathrm{Ind}_K^G \chi$, then all $|G : K|$ conjugates of χ are constituents of $\mathrm{Res}_K^G \phi$. Finally compare degrees to get the result.]

11 Construct the character table of the symmetric group S_6. Identify which of your characters are equal to the characters $\chi^{(6)}, \chi^{(5,1)}, \chi^{(4,2)}, \chi^{(3,3)}$ constructed in question 8.

12 If θ is a faithful character of the group G, which takes r distinct values on G, prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.
[Hint: assume that $\langle \chi, \theta^i \rangle = 0$ for all $i < r$; use the fact that the Vandermonde $r \times r$ matrix involving the row of the distinct values $a_1, ..., a_r$ of θ is nonsingular to obtain a contradiction.]

SM, Michaelmas Term 2020
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`