MATHEMATICAL TRIPOS, PART II, 2020/2021
REPRESENTATION THEORY EXAMPLE SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.
(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.
(b) Prove that $G/\ker\delta$ is abelian.
(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta\otimes\rho : G \to \mathrm{GL}(V)$ given by $\theta\otimes\rho : g \mapsto \theta(g)\cdot\rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

4 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G'.

5 Describe Weyl's unitary trick.

Let G be a finite group acting on a complex vector space V, and let $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ be a skew-symmetric form, i.e. $\langle\ y,\ x\ \rangle = -\langle\ x,\ y\ \rangle$ for all x, y in V.

Show that the form $(x, y) = \frac{1}{|G|}\sum\langle gx, gy\rangle$, where the sum is over all elements $g \in G$, is a G-invariant skew-symmetric form.

Does this imply that every finite subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ is conjugate to a subgroup of the symplectic group[1] $\mathrm{Sp}_{2m}(\mathbb{C})$?

6 Let $G = \langle g\rangle$ be a cyclic group of order n.
(i) G acts on $\mathbb{R}^2$ as symmetries of the regular n-gon. Choose a basis of $\mathbb{R}^2$, and write the matrix $R(g)$ representing the action of a generator g in this basis. Is this an irreducible representation?
(ii) Now regard $R(g)$ above as a complex matrix, so that we get a representation of G on $\mathbb{C}^2$. Decompose $\mathbb{C}^2$ into its irreducible summands.

7 Let G be a cyclic group of order n. Explicitly decompose the complex regular representation of G as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

[1] the group of all linear transformations of a $2m$-dimensional vector space over $\mathbb{C}$ that preserve a non-degenerate, skew-symmetric, bilinear form.

8 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y \ : \ x^5 = 1 = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

9 Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2,\ yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

10 Suppose that F is algebraically closed. Using Schur's lemma, show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H. What happens for $F = \mathbb{R}$?

11 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

12 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^tA = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

SM, Michaelmas Term 2020
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`