

## Representation Theory — Examples Sheet 2

*On this sheet all representations are complex representations*

1. Let  $(\rho, V)$  be a representation of a finite group  $G$  with character  $\chi$ . Show that  $\ker \rho = \{g \in G \mid \chi(g) = \chi(1)\}$ . Show further that  $|\chi(g)| \leq \chi(1)$  for all  $g \in G$ , with equality precisely if  $\rho(g) = \lambda \text{id}_V$  for some  $\lambda \in \mathbb{C}^\times$ . Explain how the set of normal subgroups of  $G$  may be calculated directly from the character table.
2. Let  $\chi$  be the character of a representation of a group  $G$  and let  $g \in G$ . If  $g$  has order 2 show that  $\chi(g) \in \mathbb{Z}$  and that  $\chi(g) \equiv \chi(1) \pmod{2}$ . Show that if in addition  $G$  is a non-cyclic simple group then  $\chi(g) \equiv \chi(1) \pmod{4}$ . If instead  $g$  has order 3 and is conjugate to  $g^2$  show that  $\chi(g) \equiv \chi(1) \pmod{3}$ .
3. Construct the character tables of the dihedral group  $D_8$  and the quaternion group  $Q_8$ . What do you notice? Compare the determinants of their respective two dimensional representations.
4. Construct the character tables of the dihedral groups  $D_{10}$  and  $D_{12}$ . How do the irreducible representations decompose when restricted to the subgroups of rotations?
5. Construct the character tables of  $A_4$ ,  $S_4$ ,  $A_5$  and  $S_5$ . The action of  $S_n$  on  $A_n$  by conjugation induces an action on the character table of  $A_n$  by permuting the conjugacy classes. Describe what this does to the rows of the character table for  $n = 4, 5$ .
6. Show that there is only one non-abelian group of order 21 up to isomorphism. Construct its character table.
7. A group of order 720 has 11 conjugacy classes. Two representations of the group are known and have corresponding characters  $\alpha$  and  $\beta$ . The table below summarises the sizes of the conjugacy classes and the values of  $\alpha$  and  $\beta$  on them. Prove that the group has an irreducible representation of degree 16 and calculate its character.

$  [g]  $	1	15	40	90	45	120	144	120	90	15	40
$\alpha$	6	2	0	0	2	2	1	1	0	-2	3
$\beta$	21	1	-3	-1	1	1	1	0	-1	-3	0

8. A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have characters  $\alpha, \beta$  and  $\gamma$  summarised in the table below. Construct the character table of the group. You may assume if required that  $\sqrt{7}$  is not in the field generated by  $\mathbb{Q}$  and a primitive 7<sup>th</sup> root of unity.

$  [g]  $	1	21	42	56	24	24
$\alpha$	14	2	0	-1	0	0
$\beta$	15	-1	-1	0	1	1
$\gamma$	16	0	0	-2	2	2

9. Consider the action of a finite group  $G$  by conjugation. What is the character of the corresponding permutation representation  $\mathbb{C}G$ ? Prove that the sum of elements in any row of the character table of  $G$  is a non-negative integer.
10. Show that the character table of a finite group  $G$  is invertible when viewed as a matrix.

By considering the actions induced on the rows and on the columns of the character table by complex conjugation, show that the number of irreducible characters of  $G$  that only take real values is the number of self-inverse conjugacy classes.

11. Let  $G$  be a finite group and  $\chi$  be an irreducible character of  $G$ . By beginning with the irreducible representations, show that if  $(\rho, V)$  is any representation of  $G$  then  $\frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} \rho(g)$  is a  $G$ -linear projection onto a subspace of  $V$ . Deduce that every representation can be decomposed uniquely into isotypical components.

Comments and Corrections to S.J.Wadsley@dpmms.cam.ac.uk.