

PART II REPRESENTATION THEORY
SHEET 3

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually \mathbb{C} .

1 Recall the character table of S_4 from Sheet 2. Find all the characters of S_5 induced from the irreducible characters of S_4 . Hence find the complete character table of S_5 .

Repeat, replacing S_4 by the subgroup $\langle (12345), (2354) \rangle$ of order 20 in S_5 .

2 Recall the construction of the character table of the dihedral group D_{10} of order 10 from Sheet 2.

(a) Use induction from the subgroup D_{10} of A_5 to A_5 to obtain the character table of A_5 .

(b) Let G be the subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ consisting of upper triangular matrices. Compute the character table of G .

Hint: bear in mind that there is an isomorphism $G/Z \rightarrow D_{10}$.

3 Let H be a subgroup of the group G . Show that for every irreducible representation ρ for G there is an irreducible representation ρ' for H with ρ a component of the induced representation $\mathrm{Ind}_H^G \rho'$.

Prove that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G : A|$.

4 Obtain the character table of the dihedral group D_{2m} of order $2m$, by using induction from the cyclic subgroup C_m . [Hint: consider the cases m odd and m even separately, as for m even there are two conjugacy classes of reflections, whereas for m odd there is only one.]

5 Prove the transitivity of induction: if $H < K < G$ then

$$\mathrm{Ind}_K^G \mathrm{Ind}_H^K \rho \cong \mathrm{Ind}_H^G \rho$$

for any representation ρ of H .

6 (a) Let $V = U \oplus W$ be a direct sum of $\mathbb{C}G$ -modules. Prove that both the symmetric square and the exterior square of V have submodules isomorphic to $U \otimes W$.

(b) Calculate $\chi_{\Lambda^2 \rho}$ and $\chi_{S^2 \rho}$, where ρ is the irreducible representation of dimension 2 of D_8 ; repeat this for Q_8 . Which of these characters contains the trivial character in the two cases?

7 Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G of dimension d .

(a) Compute the dimension of $S^n V$ and $\Lambda^n V$ for all n .

(b) Let $g \in G$ and let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of g on V . What are the eigenvalues of g on $S^n V$ and $\Lambda^n V$?

(c) Let $f(t) = \det(g - tI)$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of f and $\chi_{\Lambda^n V}$?

(d) Find a relationship between $\chi_{S^n V}$ and f .

8 Let G be the symmetric group S_n acting naturally on the set $X = \{1, \dots, n\}$. For any integer $r \leq \frac{n}{2}$, write X_r for the set of all r -element subsets of X , and let π_r be the permutation character of the action of G on X_r . Observe $\pi_r(1) = |X_r| = \binom{n}{r}$. If $0 \leq \ell \leq k \leq n/2$, show that

$$\langle \pi_k, \pi_\ell \rangle = \ell + 1.$$

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G, \chi^{(n-1,1)}, \chi^{(n-2,2)}, \dots, \chi^{(n-m,m)}$ such that for all $r \leq m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \dots + \chi^{(n-r,r)}.$$

In particular the class functions $\pi_r - \pi_{r-1}$ are irreducible characters of S_n for $1 \leq r \leq n/2$ and equal to $\chi^{(n-r,r)}$.

9 Let $\rho : G \rightarrow \text{GL}(V)$ be a complex representation for G affording the character χ . Give the characters of the representations $V \otimes V$, S^2V and Λ^2V in terms of χ .

(i) Let W be another finite-dimensional representation with character ψ . Show that

$$\dim W^G = \frac{1}{|G|} \sum_{g \in G} \psi(g)$$

where $W^G = \{w \in W : gw = w \text{ for all } g \in G\}$.

(ii) Prove that if V is irreducible, $V \otimes V$ contains the trivial representation at most once.

(iii) Given any irreducible character χ of G , the *indicator* $\iota\chi$ of χ is defined by

$$\iota\chi = \frac{1}{|G|} \sum_{x \in G} \chi(x^2).$$

By using the decomposition $V \otimes V = S^2V \oplus \Lambda^2V$, deduce that

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

Deduce that if $|G|$ is odd then G has only one real-valued irreducible character.

[Remark. The sign $+$, resp. $-$, indicates whether $\rho(G)$ preserves an orthogonal, respectively symplectic form on V , and whether or not the representation can be realised over the reals. You can read about it in Ch. 23 of James and Liebeck.]

10 Suppose that G is a Frobenius group with Frobenius kernel K . Show that

(i) $C_G(k) \leq K$ for all $1 \neq k \in K$.

(ii) if χ is a non-trivial irreducible character of K then $\text{Ind}_K^G \chi$ is also irreducible with K not lying in its kernel. Hence explain how to construct the character table of G , given the character tables of K and G/K .

[Hints for (ii):

(a) First, show each element of $G \setminus K$ permutes the conjugacy classes in K , and fixes only the identity.

(b) Deduce that each element of $G \setminus K$ fixes only the trivial character of K .

(c) Use the Orbit-Stabilizer theorem to deduce that if χ is a non-trivial irreducible character of K then the number of distinct conjugates of χ is $|G : K|$.

(d) Use Frobenius reciprocity to show that if χ is as above and ϕ is an irreducible constituent of $\text{Ind}_K^G \chi$, then all $|G : K|$ conjugates of χ are constituents of $\text{Res}_K^G \phi$. Finally compare degrees to get the result.]

11 Construct the character table of the symmetric group S_6 . Identify which of your characters are equal to the characters $\chi^{(6)}, \chi^{(5,1)}, \chi^{(4,2)}, \chi^{(3,3)}$ constructed in question 8.

12 If θ is a faithful character of the group G , which takes r distinct values on G , prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.
[Hint: assume that $\langle \chi, \theta^i \rangle = 0$ for all $i < r$; use the fact that the Vandermonde $r \times r$ matrix involving the row of the distinct values a_1, \dots, a_r of θ is nonsingular to obtain a contradiction.]

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