

PART II REPRESENTATION THEORY

SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually \mathbb{C} .

- 1 Let ρ be a representation of the group G .
 - (a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G .
 - (b) Prove that $G/\ker \delta$ is abelian.
 - (c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

- 2 Let $\theta : G \rightarrow F^\times$ be a 1-dimensional representation of the group G , and let $\rho : G \rightarrow \mathrm{GL}(V)$ be another representation. Show that $\theta \otimes \rho : G \rightarrow \mathrm{GL}(V)$ given by $\theta \otimes \rho : g \mapsto \theta(g) \cdot \rho(g)$ is a representation of G , and that it is irreducible if and only if ρ is irreducible.

- 3 Find an example of a representation of some finite group over some field of characteristic p , which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

- 4 Let N be a normal subgroup of the group G . Given a representation of the quotient G/N , use it to obtain a representation of G . Which representations of G do you get this way?
 Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G' .

- 5 Describe Weyl's unitary trick.
 Let G be a finite group acting on a complex vector space V , and let $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ be a skew-symmetric form, i.e. $\langle y, x \rangle = -\langle x, y \rangle$ for all x, y in V .
 Show that the form $(x, y) = \frac{1}{|G|} \sum \langle gx, gy \rangle$, where the sum is over all elements $g \in G$, is a G -invariant skew-symmetric form.
 Does this imply that every finite subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ is conjugate to a subgroup of the symplectic group¹ $\mathrm{Sp}_{2m}(\mathbb{C})$?

- 6 Let G be a cyclic group of order n .
 - (i) G acts on \mathbb{R}^2 as symmetries of the regular n -gon. Choose a basis of \mathbb{R}^2 , and write the matrix $R(1)$ representing the action of $1 \in G$ in this basis. Is this an irreducible representation?
 - (ii) Now regard $R(1)$ above as a complex matrix, so that we get a representation of G on \mathbb{C}^2 . Decompose \mathbb{C}^2 into its irreducible summands.

- 7 Let G be a cyclic group of order n . Explicitly decompose the complex regular representation of G as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g \in G}$ to a basis where the group action is diagonal.

¹the group of all linear transformations of a $2m$ -dimensional vector space over \mathbb{C} that preserve a non-degenerate, skew-symmetric, bilinear form.

- 8** Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y : x^5 = 1 = y^2, yxy^{-1} = x^{-1} \rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over \mathbb{R} .

- 9** Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over \mathbb{R} ; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

- 10** Suppose that F is algebraically closed. Using Schur's lemma, show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H . What happens for $F = \mathbb{R}$?

- 11** Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

- 12** Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over \mathbb{R} .

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Comments on and corrections to this sheet may be emailed to sm@dpmms.cam.ac.uk