Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field \( F \) of characteristic zero, usually \( \mathbb{C} \).

1. Let \( \rho \) be a representation of the group \( G \).
   (a) Show that \( \delta : g \mapsto \det \rho(g) \) is a 1-dimensional representation of \( G \).
   (b) Prove that \( G/\ker \delta \) is abelian.
   (c) Assume that \( \delta(g) = -1 \) for some \( g \in G \). Show that \( G \) has a normal subgroup of index 2.

2. Let \( \theta : G \to F^* \) be a 1-dimensional representation of the group \( G \), and let \( \rho : G \to \text{GL}(V) \) be another representation. Show that \( \theta \otimes \rho : G \to \text{GL}(V) \) given by \( \theta \otimes \rho : g \mapsto \theta(g) \cdot \rho(g) \) is a representation of \( G \), and that it is irreducible if and only if \( \rho \) is irreducible.

3. (Counterexamples to Maschke’s Theorem)
   (a) Let \( FG \) denote the regular \( FG \)-module (i.e. the permutation module coming from the action of \( G \) on itself by left multiplication), and let \( F \) be the trivial module. Find all the \( FG \)-homomorphisms from \( FG \) to \( F \) and vice versa. By considering a submodule of \( FG \) isomorphic to \( F \), prove that whenever the characteristic of \( F \) divides the order of \( G \), there is a counterexample to Maschke’s Theorem.
   (b) Find an example of a representation of some finite group over some field of characteristic \( p \), which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

4. Describe Weyl’s unitary trick.
   Let \( G \) be a finite group acting on a complex vector space \( V \), and let \( \langle \ , \ \rangle : V \times V \to \mathbb{C} \) be a skew-symmetric form, i.e. \( \langle y, x \rangle = -\langle x, y \rangle \) for all \( x, y \) in \( V \).
   Show that the form \( (x, y) = \frac{1}{|G|} \sum (gx, gy) \), where the sum is over all elements \( g \in G \), is a \( G \)-invariant skew-symmetric form.
   Does this imply that every finite subgroup of \( \text{GL}_{2m}(\mathbb{C}) \) is conjugate to a subgroup of the symplectic group \( \text{Sp}_{2m}(\mathbb{C}) \)?

5. Let \( G = \mathbb{Z}/n \) be a cyclic group of order \( n \). Explicitly decompose the (complex) regular representation of \( G \) as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis \( \{ e_g \}_{g \in G} \) to a basis where the group action is diagonal.

6. Let \( G \) be the dihedral group \( D_{10} \) of order 10,
   \[ D_{10} = \langle x, y : x^5 = 1 = y^2, yxy^{-1} = x^{-1} \rangle. \]
   Show that \( G \) has precisely two 1-dimensional representations. By considering the effect of \( y \) on an eigenvector of \( x \) show that any complex irreducible representation of \( G \) of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over \( \mathbb{R} \).
7 Let $G$ be the quaternion group $Q_8$ of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle.$$ 

By considering the effect of $y$ on an eigenvector of $x$ show that any complex irreducible representation of $G$ of dimension at least 2 is isomorphic to the standard representation of $Q_8$ of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, $Q_8$ is not a subgroup of $\text{GL}_2(\mathbb{R})$.

8 Show that if $G$ is a finite group with trivial centre and $H$ is a subgroup of $G$ with non-trivial centre, then any faithful representation of $G$ is reducible on restriction to $H$.

9 Let $G$ be a subgroup of order 18 of the symmetric group $S_6$ given by

$$G = \langle (123), (456), (23)(56) \rangle.$$ 

Show that $G$ has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that $G$ has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that $G$ has no faithful irreducible representations.

10 In this question work over the field $F = \mathbb{R}$.

Let $G$ be the cyclic group of order 3.

(a) Write the regular $\mathbb{R}G$-module as a direct sum of irreducible submodules.

(b) Find all the $\mathbb{R}G$-homomorphisms between the irreducible $\mathbb{R}G$-modules.

(c) Show that the conclusion of Schur’s Lemma (‘every homomorphism from an irreducible module to itself is a scalar multiple of the identity’) is false if you replace $\mathbb{C}$ by $\mathbb{R}$.

From now on let $G$ be a cyclic group of order $n$. Show that:

(d) If $n$ is even, the regular $\mathbb{R}G$-module is a direct sum of two (non-isomorphic) 1-dimensional irreducible submodules and $(n-2)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

(e) If $n$ is odd, the regular $\mathbb{R}G$-module is a direct sum of one 1-dimensional irreducible submodule and $(n-1)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

[Hint: use the fact that $\mathbb{R}G \subset \mathbb{C}G$ and what you know about the regular $\mathbb{C}G$-module from question 5.]

11 Show that if $\rho$ is a homomorphism from the finite group $G$ to $\text{GL}_n(\mathbb{R})$, then there is a matrix $P \in \text{GL}_n(\mathbb{R})$ such that $P \rho(g) P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix $A$ is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$. 
12 Let \( J_{\lambda,n} \) be the \( n \times n \) Jordan block with eigenvalue \( \lambda \in K \) (\( K \) is any field):

\[
J_{\lambda,n} = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & 1 & \\
0 & \cdots & \cdots & 0 & \lambda
\end{pmatrix}
\]

(a) Compute \( J_{\lambda,n}^r \) for each \( r \geq 0 \).

(b) Let \( G = \mathbb{Z}/N \) be cyclic of order \( N \), and let \( K \) be an algebraically closed field of characteristic \( p \geq 0 \). Determine all the representations of \( G \) on vector spaces over \( K \), up to equivalence. Which are irreducible?

13 A hermitian inner product on \( \mathbb{C}^2 \) is given by a \( 2 \times 2 \) matrix \( X \) such that \( \bar{x}^T X = x^T \bar{X} \); the inner product is \( \langle x, y \rangle = x^T X \bar{y} \). Explicitly find a hermitian inner product invariant under the group \( G \leq \text{GL}_2(\mathbb{C}) \) generated by the matrix

\[
\begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}
\]

[Hint: average the standard hermitian inner product.]

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Comments on and corrections to this sheet may be emailed to sm@dpmms.cam.ac.uk