PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first eight questions (but not question 6) we let $G = \mathrm{SU}(2)$. The last four questions are roughly of Tripos standard.

1 (a) Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. Describe the character χ_n of ρ_n.

(b) Decompose $V_4 \otimes V_3$ into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$. In this and the following questions, you are not being asked to find such an isomorphism explicitly.)

(c) Decompose also $V_3^{\otimes 2}$, $\Lambda^2 V_3$ and $S^2 V_3$.

(d) Show that V_n is isomorphic to its dual V_n^*.

2 Decompose $V_1^{\otimes n}$ into irreducibles.

3 Determine the character of $S^n V_1$ for $n \geq 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ for $n \geq 1$.
Decompose $S^3 V_2$ into irreducibles.

4 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by

$$A : X \mapsto A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

5 Let χ_n be the character of the irreducible representation ρ_n of G on V_n.
Show that

$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\chi_n\overline{\chi_m}\,d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi}\int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

6 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive definite) symmetric bilinear form.
(b) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbb{R}^3$, that $(A, B) \mapsto -\mathrm{tr}(AB)$ is a positive definite non-degenerate invariant bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

7 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines an G-invariant inner product on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

8 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

[Hard] Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring. Find a generator for this polynomial ring.

9 The *Heisenberg group* of order p^3 is the (non-abelian) group

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of 3×3 upper unitriangular matrices over the finite field $\mathbb{F}_p$ of p elements (p prime).

Show that G has p conjugacy classes of size 1, and $p^2 - 1$ conjugacy classes of size p.

Find p^2 characters of degree 1.

Let H be the subgroup of G comprising matrices with $a = 0$. Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Check that $V_\psi = \mathrm{Ind}_H^G \rho$ is irreducible.

Now list all the irreducible representations of G, explaining why your list is complete.

10 Recall that, up to isomorphism, there are precisely two non-abelian groups of order p^3. When $p = 2$ they are D_8 and Q_8. Suppose $p = 3$ and let H be the group of order 27 which is given by:

$$H = \langle a, b, z : a^3 = b^3 = z^3 = 1, az = za, bz = zb, b^{-1}ab = az \rangle.$$

List the conjugacy classes of H, and use Theorem 16.1 to write down the character table of H.

11 Recall Sheet 3, q.7 where we used inner products to construct some irreducible characters $\chi^{(n-r,r)}$ for S_n. Let $n \in \mathbb{N}$, and let Ω be the set of all ordered pairs (i, j) with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let $G = S_n$ act on Ω in the obvious manner (namely, $\sigma(i, j) = (\sigma i, \sigma j)$ for $\sigma \in S_n$). Let's write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action.

Prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Writing $\psi = \chi^{(n-2,1,1)}$, calculate the degree of $\chi^{(n-2,1,1)}$. Find its value on any transposition and on any 3-cycle. Returning to the character table of S_6 calculated on Sheet 3, identify the character $\chi^{(4,1,1)}$.

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Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`