

**PART II REPRESENTATION THEORY**  
**SHEET 3**

*Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field  $F$  of characteristic zero, usually  $\mathbb{C}$ .*

**1** Recall the character table of  $S_4$  from Sheet 2. Find all the characters of  $S_5$  induced from the irreducible characters of  $S_4$ . Hence find the complete character table of  $S_5$ .

Repeat, replacing  $S_4$  by the subgroup  $\langle (12345), (2354) \rangle$  of order 20 in  $S_5$ .

**2** Recall the construction of the character table of the dihedral group  $D_{10}$  of order 10 from Sheet 2.

(a) Use induction from the subgroup  $D_{10}$  of  $A_5$  to  $A_5$  to obtain the character table of  $A_5$ .

(b) Let  $G$  be the subgroup of  $\mathrm{SL}_2(\mathbb{F}_5)$  consisting of upper triangular matrices. Compute the character table of  $G$ . Hint: bear in mind that there is an isomorphism  $G/Z \rightarrow D_{10}$

**3** Let  $H$  be a subgroup of the group  $G$ . Show that for every irreducible representation  $\rho$  for  $G$  there is an irreducible representation  $\rho'$  for  $H$  with  $\rho$  a component of the induced representation  $\mathrm{Ind}_H^G \rho'$ .

Prove that if  $A$  is an abelian subgroup of  $G$  then every irreducible representation of  $G$  has dimension at most  $|G : A|$ .

**4** Obtain the character table of the dihedral group  $D_{2m}$  of order  $2m$ , by using induction from the cyclic subgroup  $C_m$ . Note that it matters whether  $m$  is odd or even.

**5** Let  $H < K < G$ . Show that

$$\mathrm{Ind}_K^G \mathrm{Ind}_H^K \rho \cong \mathrm{Ind}_H^G \rho$$

for any representation  $\rho$  of  $H$ .

**6** Calculate  $\chi_{\Lambda^2 \rho}$  and  $\chi_{S^2 \rho}$ , where  $\rho$  is the irreducible representation of dimension 2 of  $D_8$ ; repeat this for  $Q_8$ . Which of these characters contains the trivial character in the two cases?

**7** Let  $\rho : G \rightarrow \mathrm{GL}(V)$  be a representation of  $G$  of dimension  $d$ .

(a) Compute the dimension of  $S^n V$  and  $\Lambda^n V$  for all  $n$ .

(b) Let  $g \in G$  and let  $\lambda_1, \dots, \lambda_d$  be the eigenvalues of  $g$  on  $V$ . What are the eigenvalues of  $g$  on  $S^n V$  and  $\Lambda^n V$ ?

(c) Let  $f(x) = \det(g - xI)$  be the characteristic polynomial of  $g$  on  $V$ . Describe how to obtain the trace  $\chi_{\Lambda^n V}(g)$  from the coefficients of  $f(x)$ .

(d)\* Find a relation between  $\chi_{S^n V}(g)$  and the polynomial  $f(x)$ . [Hint: do the case where  $V$  has dimension 1 first.]

**8** Let  $G$  be the symmetric group  $S_n$  acting naturally on the set  $X = \{1, \dots, n\}$ . For any integer  $r \leq \frac{n}{2}$ , write  $X_r$  for the set of all  $r$ -element subsets of  $X$ , and let  $\pi_r$  be the permutation character of the action of  $G$  on  $X_r$ . Observe  $\pi_r(1) = |X_r| = \binom{n}{r}$ . If  $0 \leq \ell \leq k \leq n/2$ , show that

$$\langle \pi_k, \pi_\ell \rangle = \ell + 1.$$

Let  $m = n/2$  if  $n$  is even, and  $m = (n-1)/2$  if  $n$  is odd. Deduce that  $S_n$  has distinct irreducible characters  $\chi^{(n)} = 1_G, \chi^{(n-1,1)}, \chi^{(n-2,2)}, \dots, \chi^{(n-m,m)}$  such that for all  $r \leq m$ ,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \dots + \chi^{(n-r,r)}.$$

In particular the class functions  $\pi_r - \pi_{r-1}$  are irreducible characters of  $S_n$  for  $1 \leq r \leq n/2$  and equal to  $\chi^{(n-r,r)}$ .

**9** If  $\rho : G \rightarrow \text{GL}(V)$  is an irreducible complex representation for  $G$  affording character  $\chi$ , find the characters of the representation spaces  $V \otimes V$ ,  $\text{Sym}^2(V)$  and  $\Lambda^2(V)$ .

Define the *Frobenius-Schur indicator*  $\iota\chi$  of  $\chi$  by

$$\iota\chi = \frac{1}{|G|} \sum_{x \in G} \chi(x^2)$$

and show that

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

[Remark. The sign  $+$ , resp.  $-$ , indicates whether  $\rho(G)$  preserves an orthogonal, respectively, symplectic form on  $V$ , and whether or not the representation can be realised over the reals. You can read about it in Isaacs or in James and Liebeck.]

**10** The group  $G \times G$  acts on  $G$  by  $(g, h)(x) = gxh^{-1}$ . In this way, the regular representation space  $\mathbb{C}G$  becomes a  $G \times G$ -space. (So far, we only considered  $\mathbb{C}G$  as a representation space of the group  $G \times \{1\} \leq G \times G$ .)

Determine the character  $\pi$  of  $G \times G$  in this action. For each irreducible character  $\chi\psi$  of  $G \times G$ , determine its multiplicity in  $\pi$ . Compare  $\pi$  to the character of the subgroup  $G \times \{1\}$  in this action.

**11** If  $\theta$  is a faithful character of the group  $G$ , which takes  $r$  distinct values on  $G$ , prove that each irreducible character of  $G$  is a constituent of  $\theta$  to power  $i$  for some  $i < r$ .

[Hint: assume that  $\langle \chi, \theta^i \rangle = 0$  for all  $i < r$ ; use the fact that the Vandermonde  $r \times r$  matrix involving the row of the distinct values  $a_1, \dots, a_r$  of  $\theta$  is nonsingular to obtain a contradiction.]

**12** Construct the character table of the symmetric group  $S_6$ . Identify which of your characters are equal to the characters  $\chi^{(6)}, \chi^{(5,1)}, \chi^{(4,2)}, \chi^{(3,3)}$  constructed in question 7.

SM, Lent Term 2010

Comments on and corrections to this sheet may be emailed to [sm@dpmms.cam.ac.uk](mailto:sm@dpmms.cam.ac.uk)