PART II REPRESENTATION THEORY
SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.
(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.
(b) Prove that $G/\ker \delta$ is abelian.
(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta \otimes \rho : G \to \mathrm{GL}(V)$ given by $\theta \otimes \rho : g \mapsto \theta(g) \cdot \rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 Given any prime p. Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group. [Thus Maschke's Theorem can fail if F is not $\mathbb{R}$ or $\mathbb{C}$ or if G is not finite.]

4 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from the abelianisation G/G'.

5 Describe Weyl's unitary trick.

Let G be a finite group acting on a complex vector space V, and let $\langle\ ,\ \rangle$ be an alternating bilinear form from $V \times V$ to $\mathbb{C}$ (so $\langle\, y,\ x\,\rangle = -\langle\, x,\ y\,\rangle$ for x, y in V).

Show that the form $(x, y) = \frac{1}{|G|} \sum \langle gx, gy \rangle$, where the sum is over all elements $g \in G$, is a G-invariant alternating form.

Does this imply that every finite subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ is conjugate to a subgroup of the symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$?

6 Let G be a cyclic group of order n. Decompose the regular representation of G explicitly as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

7 Let G be the dihedral group D_{10} of order 10, with presentation

$$D_{10} = \langle x, y \ :\ x^5 = 1 = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

8 Let G be the quaternion group with presentation

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2,\ yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

9 State Maschke's theorem.

Show that any irreducible complex representation of the finite group G is isomorphic to a subrepresentation of the regular representation of G.

10 State Schur's lemma.

Show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

11 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

12 Work over $F = \mathbb{R}$. Show that the cyclic group $C_3 = \mathbb{Z}/3$ has up to equivalence only one non-trivial irreducible representation over $\mathbb{R}$. If (ρ, V) is this representation, show that $\dim_{\mathbf{R}} \mathrm{Hom}_G(V, V) = 2$. Comment.

13 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

14 Prove that for every finite simple group G, there exists a faithful irreducible complex representation. (Hint: recall that the regular representation is faithful).

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Comments and corrections on this sheet may be emailed to `sm@dpmms.cam.ac.uk`