Part II Representation Theory Sheet 1

Unless otherwise stated, groups here are finite, and all vector spaces are finite dimensional over a field F of characteristic zero, usually $\mathbf{C}$.

Q.1 Let ρ be a representation of the group G. Show that det ρ is a one-dimensional representation of G.

Q.2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, let $\rho : G \to GL(V)$ be another representation. Show that $\theta \otimes \rho : G \to GL(V)$ given by $\theta \otimes \rho : g \mapsto \theta(g) \cdot \rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

Q.3 Let $\rho : \mathbf{Z} \to GL(2, \mathbf{C})$ be the matrix representation defined by $\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that ρ is not completely reducible.

Q.4 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G'.

Q.5 Describe Weyl's unitary trick.

Let G be a finite group acting on a complex vector space V, and let $\langle\ ,\ \rangle$ be an alternating bilinear form from $V \times V$ to $\mathbf{C}$ (so $\langle\, y,\ x\, \rangle = -\langle\, x,\ y\, \rangle$ for x, y in V).

Show that the form $(x, y) = |G|^{-1} \sum \langle gx, gy \rangle$, where the sum is over all elements $g \in G$, is a G-invariant alternating form.

Does this imply that every finite subgroup of $GL(2m, \mathbf{C})$ is conjugate to a subgroup of the symplectic group $Sp(2m, \mathbf{C})$?

Q.6 Let G be a cyclic group of order n. Decompose the regular representation of G explicitly as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g \in G}$ to a basis where the group action is diagonal.

Q.7 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y \mid x^5 = 1 = y^2, yxy^{-1} = x^{-1} \rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbf{R}$.

Q.8 Let G be the quaternion group

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbf{R}$; that is, Q_8 is not a subgroup of $GL(2, \mathbf{R})$.

Q.9 State Maschke's theorem.

Show that any irreducible complex representation of the finite group G is isomorphic to a subrepresentation of the regular representation of G.

Q.10 State Schur's lemma.

Show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

Q.11 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2, none of which is faithful. It follows that G has no faithful irreducible representations.

Q.12 Work over $F = \mathbf{R}$. Show that the cyclic group $C_3 = \mathbf{Z}/3$ has up to equivalence only one non-trivial irreducible representation over $\mathbf{R}$. If (ρ, V) is this representation, show that $\dim_{\mathbf{R}} \mathrm{Hom}_G(V, V) = 2$. Comment.

Q.13 Show that if ρ is a homomorphism from the finite group G to $GL(n, \mathbf{R})$, then there is a matrix $P \in GL(n, \mathbf{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbf{R}$.