

II Representation Theory Sheet 3

Unless otherwise stated, groups here are finite, all vector spaces are finite dimensional over a field F of characteristic zero, usually \mathbf{C} .

Q.1 Find all the characters of S_5 induced by the irreducible characters of S_4 . Hence find the character table of S_5 .

Repeat, replacing S_4 by the subgroup $\langle (12345), (2354) \rangle$ of order 20 in S_5 .

Q.2 Construct the character table of the dihedral group D_{10} of order 10. Use induction from the subgroup D_{10} of A_5 to A_5 to obtain the character table of A_5 .

Q.3 Let H be a subgroup of the group G . Show that for every irreducible representation ρ for G there is an irreducible representation ρ' for H with ρ a component of the induced representation $\text{Ind}_H^G \rho'$.

Prove that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G : A|$.

Q.4 Obtain the character table of the dihedral group D_{2m} of order $2m$, by using induction from the cyclic subgroup C_m . Note that it matters whether m is odd or even.

Q.5 Prove the transitivity of induction: given $H < K < G$, with ρ a representation of H , show that $\text{Ind}_K^G \text{Ind}_H^K \rho = \text{Ind}_H^G \rho$.

Q.6 Calculate $\chi_{\Lambda^2 \rho}$ and $\chi_{S^2 \rho}$, where ρ is the irreducible representation of dimension 2 of D_8 , and repeat for Q_8 . Which of these characters contains the principal character in the two cases?

Q.7 Let $\rho : G \rightarrow GL(V)$ be a representation of G of dimension d .

i) Compute the dimension of $S^n V$ and $\Lambda^n V$ for all n .

ii) Let $g \in G$ and let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of g on V . What are the eigenvalues of g on $S^n V$ and $\Lambda^n V$?

iii) Let $f(x) = \det(g - xI)$ be the characteristic polynomial of g on V . Describe how to obtain $\chi_{\Lambda^n V}(g)$ from the coefficients of $f(x)$.

iv*) Find a relation between $\chi_{S^n V}(g)$ and the polynomial $f(x)$.

[Do the case where V has dimension 1 first.]

Q.8 (i) Let G be the symmetric group S_n , let $X = \{1, \dots, n\}$. Write X_r for the set of all r -element subsets of X , and let π_r be the permutation character of the action of G on X_r . If $r \leq s \leq n/2$, show that G has $r + 1$ orbits in its action on $X_r \times X_s$, and deduce that $\langle \pi_r, \pi_s \rangle = r + 1$. Deduce that the generalized character $\pi_r - \pi_{r-1}$ is an irreducible character for $1 \leq r \leq n/2$.

(ii) Repeat with G the general linear group $GL(X)$, where X is a vector space of dimension n over a finite field, and X_r is the set of all r -dimensional subspaces of X .

Q.9 If $\rho : G \rightarrow GL(V)$ is an irreducible complex representation for G affording character χ , find the characters of the representation spaces $V \otimes V$, $Sym^2(V)$ and $\Lambda^2(V)$.

Deduce that

$$\frac{1}{|G|} \sum_{x \in G} \chi(x^2) = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued;} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

[Remark. The sign +, resp - , indicates whether $\rho(G)$ preserves an orthogonal, resp symplectic form on V , and whether or not the representation can be realized over the reals. You can read about it in Isaacs or in James and Liebeck - it is the Frobenius Schur indicator.]

Q.10 The group $G \times G$ acts on G by $(g, h)(x) = gxh^{-1}$. In this way, the regular representation space $\mathbf{C}[G]$ becomes a $G \times G$ -space. (So far, we only considered $\mathbf{C}[G]$ as a representation space of the group $G \times \{1\} \leq G \times G$.)

Determine the character π of $G \times G$ in this action. For each irreducible character $\chi\psi$ of $G \times G$, determine its multiplicity in π . Compare π to the character of the subgroup $G \times \{1\}$ in this action.

Q.11 If θ is a faithful character of the group G , which takes r distinct values on G , prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.

[Assume that $\langle \chi, \theta^i \rangle = 0$ for all $i < r$; use the fact that the Vandermonde $r \times r$ matrix involving the row of the distinct values a_1, \dots, a_r of θ is nonsingular to obtain a contradiction.]

Q.12 Construct the character table of the symmetric group S_6 .