Representation Theory IIB, Sheet 1

IG, 2006

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G is a group; vector spaces are finite-dimensional and the field of scalars is $\mathbb{C}$, unless otherwise stated.

1.1 Question

Let ρ be a representation of G. Show that $\det \rho$ is a 1-dimensional representation of G.

1.2 Question

Let $\rho : \mathbb{Z} \to GL(2;\mathbb{C})$ be the representation of $\mathbb{Z}$ defined by $\rho(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that ρ is not completely reducible.

1.3 Question

Let $\theta : G \to \mathbb{C}^\times$ be a one-dimensional representation of G, and let ρ be another representation on a vector space V.
Show that $\theta \otimes \rho : G \to GL(V)$, $(\theta \otimes \rho)(g) := \theta(g) \cdot \rho(g)$ is also a representation, and is irreducible if and only if ρ is.

1.4 Question

Let N be a normal subgroup of G. Given a representation of G/N, define a representation of G, and describe which representations of G arise in this way.

1.5 Question

(a) *(Weyl's unitary trick)* Let V be a representation of the finite group G and let $\langle\,|\,\rangle$ be any inner product on V. Show that the averaged inner product $\langle\,|\,\rangle'$ defined by

$$\langle \mathbf{v}|\mathbf{w}\rangle' = \frac{1}{|G|}\sum\nolimits_{g\in G}\langle g\mathbf{v}|g\mathbf{w}\rangle$$

is G-invariant.
(b) Can you conclude by a similar argument that every even-dimensional representation of G carries an invariant, non-degenerate *skew-symmetric* bilinear form?
(A skew-symmetric form satisfies $\langle \mathbf{v}|\mathbf{w}\rangle = -\langle \mathbf{w}|\mathbf{v}\rangle$, and it is non-degenerate if for any $\mathbf{v} \in V \setminus \{0\}$ there exists $\mathbf{w} \in V$ with $\langle \mathbf{v}|\mathbf{w}\rangle \neq 0$.)

1.6 Question

Let G be a cyclic group of order n. Decompose the regular representation explicitly as a sum of one-dimensional representation, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

1.7 Question

Let X be a finite set with G-action and ρ_X the associated permutation representation (on the vector space $\mathbb{C}[X]$, with basis $\{\mathbf{e}_x\}_{x\in X}$. Show that the value at $g \in G$ of the character of ρ_X is the number of fixed points of g in X.

1.8 Question

a) Show that if $\rho : G \to GL_n(\mathbb{R})$ is a homomorphism, then there exists a matrix $P \in GL_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that A is orthogonal if $A^T A = I$).
b) Determine all finite groups which have a faithful representation on a two dimensional *real* vector space.

1.9 Question

Let $G = \mathbb{Z}/M \times \mathbb{Z}/N$. Determine all the irreducible complex representations of G.

1.10 Question

Let $G = \mathbb{Z}/N$. G acts on $\mathbb{R}^2$ as rotations of the plane. Choose a basis of $\mathbb{R}^2$, and write the matrix $\rho(1)$ representing the action of $1 \in \mathbb{Z}/N$ in this basis.
a) Is this an irreducible representation?
b) Now regard $\rho(1)$ as a complex matrix, so that this defines a representation on $\mathbb{C}^2$. Decompose this into irreducible summands.

1.11 Question

A A 2-by-2 matrix X such that $\overline{X}^T = X$ defines a hermitian inner product on $\mathbb{C}^2$ by $\langle x, y\rangle = x^T X \overline{y}$. Explicitly find a Hermitian inner product invariant under the group G generated by the matrix $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$.

1.12 Question

Do all the exercises set in lectures!!

1.13 Question

Let G be a finite abelian group and denote by $\widehat{G}$ the set of isomorphism classes of irreducible G-representations.
(a) Show that $\widehat{G}$ forms an abelian group, under the tensor product of representations (as in Question 2)
(b) If G is cyclic of order n, what can you say about $\widehat{G}$?
(c) Fix $g \in G$. Show that the map $\widehat{G} \to \mathbb{C}^\times$ sending a representation χ to the value of $\chi(g)$ is a group homomorphism, and conclude that this assignment defines a homomorphism $G \to \widehat{\widehat{G}}$
(d) Show that this homomorphism is an isomorphism.
(e) Show that a homomorphism $\phi : G \to H$ of abelian groups induces a homomorphism $\widehat{\phi} : \widehat{H} \to \widehat{G}$, by sending $\chi : H \to \mathbb{C}^\times$ to $\widehat{\chi} := \chi \circ \phi$, and that $\widehat{\phi}$ is surjective iff ϕ is injective.