Representation Theory Sheet 4

CT, Lent 2005

Starred questions are optional. For help with Questions 1 and 2, consult the appropriate section of the notes.

4.1 Question

Show that any continuous group homomorphisms $\phi : U(1) \to \mathbb{C}^\times$ lands in the subgroup $U(1)$ of unit norm complex numbers. Moreover, show that: either $\phi \equiv 1$, or else $\ker \phi$ is a cyclic subgroup of roots of unity.

4.2 Question

Let ϕ be as in Question 1, with kernel of order n. Show that $\phi(z) = z^n$ or $\phi(z) = z^{-n}$, as follows.
Define $f : (0, 2\pi/n) \to (0, 2\pi)$ by $f(\theta) = \arg \phi(e^{i\theta})$. Show that f is monotonically increasing or decreasing.
Show that $f(x) = nx$ or $f(x) = -nx$, for all rational numbers x.
[*Use the fact that ϕ is a homomorphism, is monotonic, and compare the number of points of appropriate orders in the two groups $U(1)$.*]

Finish the proof by continuity.

4.3 Question

Prove that the left×right action of $SU(2)\times SU(2)$ on quaternions defines an isomorphism $SU(2)\times SU(2)/\{\pm(1,1)\} \cong SO(4)$. (Given the work in class, you must only check surjectivity of the map.)

4.4 Question

Let $G = SU(2)$ and let S_n be its irreducible $(n+1)$-dimensional representation. (So $S_0 = 1$). Using the character formulae for the S_n, decompose the following representations into irreducibles:

(a) $\mathrm{Sym}^2 S_3$; (b) $\Lambda^2 S_3$; (c) $S_1^{\otimes n}$; (d) $\mathrm{Sym}^3 S_2$.

[*For part (d), refer to Sheet 2 Q.10 or to Question 6 below.*]

4.5 Question

Describe the representation ring of $U(2)$.

4.6 Question*: Characters of symmetric and exterior powers

Let the finite or compact group G act on V.
(a) By considering the action of a group element g in an eigen-basis, with eigenvalues $\lambda_1, \dots, \lambda_d$, show that the generating function for the characters of g on the exterior powers $\Lambda^n V$ satisfies

$$\sum_{n\geq 0} t^n \chi_{\Lambda^n V}(g) = \prod_{i=1}^{d} (1 + t\lambda_i)\,,$$

while on the symmetric powers $\mathrm{Sym}^n V$,

$$\sum_{n\geq 0} t^n \chi_{\mathrm{Sym}^n V}(g) = \prod_{i=1}^{d} \frac{1}{(1 - t\lambda_i)}.$$

Express these generating series in terms of the characteristic polynomial for the action of g on V.
(b) For a function φ on the group G, define $\Psi^k[\varphi]$ to be the function $g \mapsto \varphi(g^k)$. Rewriting the right-hand sides in

the identities above as exp log of the same, show that

$$\sum_{n\geq 0} t^n \chi_{\mathrm{Sym}^n V} = \exp\left\{\sum_{n>0} t^n \Psi^n[\chi]/n\right\}, \qquad \sum_{n\geq 0}(-t)^n \chi_{\Lambda^n V} = \exp\left\{-\sum_{n>0} t^n \Psi^n[\chi]/n\right\}.$$

(c) Show, by comparing bases, that there is a natural isomorphism

$$\mathrm{Sym}^n(V \oplus W) \cong \bigoplus_{p+q=n} \mathrm{Sym}^p(V) \otimes \mathrm{Sym}^q(W),$$

and similarly for the exterior powers. (By convention, $\mathrm{Sym}^0 V = \Lambda^0(V) = \mathbb{C}$). Assemble these to the natural isomorphism of formal power series

$$\sum_{n\geq 0} t^n \mathrm{Sym}^n(V \oplus W) = \sum_{n\geq 0} t^n \mathrm{Sym}^n(V) \otimes \sum_{n\geq 0} t^n \mathrm{Sym}^n(W).$$

Splitting V into d eigen-lines for the action of g, discuss this in relation to the formula in Part (a).

4.7 Question*: the Adams operations Ψ^k

For the character χ of the representation V of G, let $s^k[\chi]$ and $\lambda^k[\chi]$ denote the characters of the kth symmetric and exterior powers of V. Refer to Question 4.6.b for the definition of $\Psi^k[\chi]$.
(a) By taking $\frac{d}{dt}$ of log of both sides in 4.6.b and equating coefficients of t-powers, conclude that

$$\sum_{k=1}^{n} \Psi^k[\chi] s^{n-k}[\chi] = n \cdot s^n[\chi], \qquad \sum_{k=1}^{n} (-1)^{k-1} \Psi^k[\chi] \lambda^{n-k}[\chi] = n \cdot \lambda^n[\chi].$$

(b) From (a), conclude by induction that $\Psi^n[\chi]$ is an *integral* linear combination of characters of G.
(c) Assume now that G is finite and n is prime to $|G|$. Show that the map $g \mapsto g^n$ is a bijection of G with itself.
(d) By calculating the norm, show that $\Psi^n[\chi]$ is irreducible, if χ is so and n is prime to $|G|$.
(e) Find the action of Ψ^n on your favourite character tables.

4.8 Question*

Show that the character of the symmetric power $\mathrm{Sym}^k(\mathbb{C}^n)$ of the standard n-dimensional representation of $U(n)$ is the Schur function $S_{(k,0,\ldots,0)}$.
[*Hint: Consider the generating function for the* Sym^k *as in Question 6a, and play with determinants.*]