Representation Theory Sheet 3

CT, Lent 2005

G is a finite group and vector spaces are finite-dimensional over $\mathbb{C}$.

3.1 Question

Prove that the inverse of the isomorphism

$$\mathbb{C}[G] \to \bigoplus_V \mathrm{End}(V), \quad \varphi \mapsto \left(\sum\nolimits_{g\in G} \varphi_g \cdot \rho_V(g)\right)$$

sends $\phi \in \mathrm{End}(V)$ to the function $g \mapsto \frac{1}{\dim V}\mathrm{Tr}_V(\phi\rho_V(g))$.
Hint: Show that this map commutes with the two actions of G and use Schur's Lemma. Checking normalisation with $\phi = \mathrm{Id}$.

3.2 Question

Let p, q be two primes with q dividing $p-1$. Let also u be a no-zero residue class mod p of multiplicative order q. Prove that the group generated by a, b subject to $a^p = b^q = 1$, $bab^{-1} = a^u$ is independent of u, up to isomorphism. If you are brave, prove that it is the only non-cyclic group of order pq.

3.3 Question

Spell out induction of representations from C_n to D_{2n}. Confirm Mackey's criterion for irreducibility, where appropriate. Same for the group $F_{p,q}$ of Q. 2 and its cyclic subgroup generated by a.

3.4 Question

Prove the transitivity of induction: if $K \subset H \subset G$ are subgroups, then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H = \mathrm{Ind}_K^G$.

3.5 Question

We have defined induction of class functions from a subgroup H to G as

$$\mathrm{Ind}_H^G(\varphi)(u) = \frac{1}{|H|} \sum_{\substack{g\in G \\ gug^{-1}\in H}} \varphi(gug^{-1}).$$

Show that this linear map Ind, from class functions on H to class functions on G, is the *hermitian adjoint* of the restriction of class functions, from G to H:

$$\langle \varphi | \mathrm{Ind}\psi \rangle_G = \langle (\varphi|_H) | \psi \rangle_H,$$

in the natural inner product $\langle \varphi|\psi\rangle_G = \frac{1}{|G|}\sum_{g\in G} \bar{\varphi}(g)\psi(g)$ (similarly for H). $\varphi|_H$ denotes the restriction to H. Use this to give another proof of the Frobenius reciprocity formula.

3.6 Question

Prove Mackey's restriction formula using characters.

3.7 Question

With reference to the character tables (2.12, 2.13), describe the induction of irreducible representations from A_4 to S_4 and to S_5, from A_5 to S_5 and from S_4 to S_5.

3.8 Question

Show that the subgroup of A_5 generated by (12345) and $(25)(34)$ is isomorphic to D_{10}, and describe the induction of irreducible representations from this D_{10} to A_5.

3.9 Question

If V is an irreducible complex representation of G with character χ, use the formulae for the characters of $V^{\otimes 2}$, $\mathrm{Sym}^2 V$ and $\Lambda^2 V$ to show that

$$\frac{1}{|G|}\sum_{u\in G}\chi(u^2) = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued;} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

Hint: Describe the subspace of G-invariant vectors in $V \otimes V$, with reference to Schur's Lemma. Now look at Question 2.8.c (Sheet 2).

3.10 Question

If you have the time, construct the character table of S_6.