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Probability and Measure 2

3.1. Suppose that a simple function f has two representations

$$f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{j=1}^{n} b_j 1_{B_j}.$$

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \ldots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly B_δ for $\delta \in \{0,1\}^n$. Then set $f_{\varepsilon,\delta} = \sum_{k=1}^m \varepsilon_k a_k$ if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon,\delta} = 0$ otherwise. Show that, for any measure μ,

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon,\delta} f_{\varepsilon,\delta} \mu(A_\varepsilon \cap B_\delta)$$

and deduce that

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j).$$

3.2. Let μ and ν be finite Borel measures on $\mathbb{R}$. Let f be a continuous bounded function on $\mathbb{R}$. Show that f is integrable with respect to μ and ν. Show further that, if $\mu(f) = \nu(f)$ for all such f, then $\mu = \nu$.

3.3. Let f be an integrable function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that, for some π-system $\mathcal{A}$ containing E and generating $\mathcal{E}$, we have $\mu(f 1_A) = 0$ for all $A \in \mathcal{A}$. Show that $f = 0$ a.e.

3.4. Let X be a non-negative integer-valued random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Deduce that, if $\mathbb{E}(X) = \infty$ and $X_1, X_2, \ldots$ is a sequence of independent random variables with the same distribution as X, then, almost surely, $\limsup_n (X_n/n) \geq 1$, and moreover $\limsup_n (X_n/n) = \infty$.

Now suppose that $Y_1, Y_2, \ldots$ is any sequence of independent identically distributed random variables with $\mathbb{E}|Y_1| = \infty$. Show that, almost surely, $\limsup_n (|Y_n|/n) = \infty$, and moreover $\limsup_n (|Y_1 + \cdots + Y_n|/n) = \infty$.

3.5. For $\alpha \in (0, \infty)$ and $x \in (0, \infty)$, define $f_\alpha(x) = x^{-\alpha}$. Show that f_α is integrable with respect to Lebesgue measure on $(0, 1]$ if and only if $\alpha < 1$. Show also that f_α is integrable with respect to Lebesgue measure on $[1, \infty)$ if and only if $\alpha > 1$.

3.6. Show that the function $\sin x/x$ is not Lebesgue integrable over $[1,\infty)$ but that integral $\int_1^N (\sin x/x)dx$ converges as $N \to \infty$.

3.7. Show that, as $n \to \infty$,

$$\int_0^\infty \sin(e^x)/(1+nx^2)dx \to 0 \quad \text{and} \quad \int_0^1 (n\cos x)/(1+n^2x^{\frac{3}{2}})dx \to 0.$$

3.8. Let u and v be differentiable functions on $\mathbb{R}$ with continuous derivatives u' and v'. Suppose that uv' and $u'v$ are integrable on $\mathbb{R}$ and $u(x)v(x) \to 0$ as $|x| \to \infty$. Show that

$$\int_{\mathbb{R}} u(x)v'(x)dx = -\int_{\mathbb{R}} u'(x)v(x)dx.$$

3.9. Let $(E,\mathcal{E})$ and $(G,\mathcal{G})$ be measurable spaces and let $f : E \to G$ be a measurable function. Given a measure μ on $(E,\mathcal{E})$, consider the image measure $\nu = \mu \circ f^{-1}$ on $(G,\mathcal{G})$. Show that $\nu(g) = \mu(g \circ f)$ for all non-negative measurable functions g on G.

3.10. The moment generating function ϕ of a real-valued random variable X is defined by $\phi(\theta) = \mathbb{E}(e^{\theta X}), \quad \theta \in \mathbb{R}$.

Suppose that ϕ is finite on an open interval containing 0. Show that ϕ has derivatives of all orders at 0 and that X has finite moments of all orders given by

$$\mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n \bigg|_{\theta=0} \phi(\theta).$$

3.11. Let $X_1,\dots,X_n$ be random variables with density functions $f_1,\dots,f_n$ respectively. Suppose that the $\mathbb{R}^n$-valued random variable $X = (X_1,\dots,X_n)$ also has a density function f. Show that $X_1,\dots,X_n$ are independent if and only if

$$f(x_1,\dots,x_n) = f_1(x_1)\dots f_n(x_n) \qquad \text{a.e.}$$

3.12. Show that, for all non-negative measurable functions f on $[0,\infty)$, the function $(x,y) \mapsto f(|(x,y)|)$ is measurable on $\mathbb{R}^2$ and (without using the Jacobian formula)

$$\int_{\mathbb{R}^2} f(|(x,y)|)dxdy = 2\pi \int_0^\infty rf(r)dr.$$

Hence show that $(2\pi)^{-1/2}e^{-x^2/2}$ is a probability density function.

3.13. Let μ and ν be probability measures on $(E,\mathcal{E})$ and let $f : E \to [0,R]$ be a measurable function. Suppose that $\nu(A) = \mu(f1_A)$ for all $A \in \mathcal{E}$. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables in E with law μ and let $(U_n : n \in \mathbb{N})$ be a sequence of independent $U[0,1]$ random variables. Set

$$T = \min\{n \in \mathbb{N} : RU_n \leq f(X_n)\}, \qquad Y = X_T.$$

Show that Y has law ν. (This justifies simulation by rejection sampling.)