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Probability and Measure 1

1.1. Let E be a set and let $\mathcal{S}$ be a set of σ-algebras on E. Define

$$\mathcal{E}^* = \{A \subseteq E : A \in \mathcal{E} \quad \text{for all} \quad \mathcal{E} \in \mathcal{S}\}.$$

Show that $\mathcal{E}^*$ is a σ-algebra on E. Show, on the other hand, by example, that the union of two σ-algebras on the same set need not be a σ-algebra.

1.2. Show that the following sets of subsets of $\mathbb{R}$ all generate the same σ-algebra:

(a) $\{(a,b) : a < b\}$, (b) $\{(a,b] : a < b\}$, (c) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.3. Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4. Show that a collection of sets $\mathcal{A}$ is a d-system if and only if the following holds:

- $\emptyset \in \mathcal{A}$
- if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
- if $(A_n : n \in \mathbb{N})$ is a collection of disjoint sets in $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Show that a π-system which is also a d-system is a σ-algebra.

1.5. Let μ be a finite-valued additive set function on a ring $\mathcal{A}$. Show that μ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $\mathcal{A}$, with $\cap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6. Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$,

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events A_n are independent if and only if the σ-algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8. Let B be a Borel subset of the interval $[0,1]$. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \triangle B$ $(= (A^c \cap B) \cup (A \cap B^c))$ is less than ε. Show further that this remains true for every Borel set in $\mathbb{R}$ of finite Lebesgue measure.

1.9. Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ-algebra and show that μ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the *completion of* $\mathcal{E}$ *with respect to* μ. Suppose now that E is σ-finite and write μ^* for the outer measure associated to μ, as in the proof of Carathéodory's Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of μ^*-measurable sets.

2.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on a measurable space $(E, \mathcal{E})$. Show that the following functions are also measurable: $f_1 + f_2$, $f_1 f_2$, $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show also that $\{x \in E : f_n(x) \text{ converges as } n \to \infty\} \in \mathcal{E}$.

2.2. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, let μ be a measure on $\mathcal{E}$, and let $f : E \to G$ be a measurable function. Show that we can define a measure ν on $\mathcal{G}$ by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$.

2.3. Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$ for all $x, y \in \mathbb{R}$.

2.4. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

2.5. The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^\infty n^{-s}$. Let X and Y be independent random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

2.6. Let $(X_n : n \in \mathbb{N})$ be independent $N(0,1)$ random variables. Prove that

$$\limsup_n \left(X_n/\sqrt{2 \log n}\right) = 1 \quad \text{a.s.}$$

2.7. Let C_n denote the nth approximation to the Cantor set C: thus $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n. Show that

(a) C is uncountable and has Lebesgue measure 0,
(b) for all $x \in [0,1]$, the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists,
(c) the function F is continuous on $[0,1]$, with $F(0) = 0$ and $F(1) = 1$,
(d) for almost all $x \in [0,1]$, F is differentiable at x with $F'(x) = 0$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.