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Probability and Measure 4

1. Let μ and ν be probability measures on $(E, \mathcal{E})$ and let $f : E \to [0, R]$ be a measurable function. Suppose that $\nu(A) = \mu(f 1_A)$ for all $A \in \mathcal{E}$. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables in E with law μ and let $(U_n : n \in \mathbb{N})$ be a sequence of independent $U[0,1]$ random variables. Set

$$T = \min\{n \in \mathbb{N} : RU_n \le f(X_n)\}, \qquad Y = X_T.$$

Show that Y has law ν. (This justifies simulation by rejection sampling.)

2. A *stepfunction* $f : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$: that is, for all $f \in L^p(\mathbb{R})$ and all $\varepsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \varepsilon$. Deduce that the set of continuous functions of compact support is also dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

3. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2} \max_{k \le n} |X_k| \to 0$ in probability.

4. Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

5. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that

$$\mathbb{E}\big(\max_{k \le n} |X_k|\big)/\sqrt{n} \to 0 \qquad \text{as} \quad n \to \infty.$$

6. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

7. Let $X = (X_1, \ldots, X_n)$ be a Gaussian random variable in $\mathbb{R}^n$ with mean μ and covariance matrix V. Assume that V is invertible write $V^{-1/2}$ for the positive-definite square root of V^{-1}. Set $Y = (Y_1, \ldots, Y_n) = V^{-1/2}(X - \mu)$. Show that $Y_1, \ldots, Y_n$ are independent $N(0,1)$ random variables. Show further that we can write X_2 in the form $X_2 = aX_1 + Z$ where Z is independent of X_1 and determine the distribution of Z.

8. Let $X_1, \ldots, X_n$ be independent $N(0,1)$ random variables. Show that

$$\left(\overline{X}, \sum_{m=1}^{n} (X_m - \overline{X})^2\right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2\right)$$

have the same distribution, where $\overline{X} = (X_1 + \cdots + X_n)/n$.

9. Call a sequence of random variables $(X_n : n \in \mathbb{N})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *stationary* if for each $n, k \in \mathbb{N}$ the random vectors $(X_1, \dots, X_n)$ and $(X_{k+1}, \dots, X_{k+n})$ have the same distribution: for $A_1, \dots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \dots, X_{k+n} \in A_n).$$

Show that, if $(X_n : n \in \mathbb{N})$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to X \qquad \text{a.s. and in } L^p,$$

for some random variable $X \in L^p$ and find $\mathbb{E}(X)$.

10. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $\mathbb{E}(X_n) = \mu$ and $\mathbb{E}(X_n^4) \leq M$ for all n, for some constants $\mu \in \mathbb{R}$ and $M < \infty$. Set $P_n = X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n$. Show that P_n / n converges a.s. as $n \to \infty$ and identify the limit.

11. The Cauchy distribution has density function $f(x) = \pi^{-1}(1 + x^2)^{-1}$ for $x \in \mathbb{R}$. Show that the corresponding characteristic function is given by $\varphi(u) = e^{-|u|}$. Show also that, if $X_1, \dots, X_n$ are independent Cauchy random variables, then the random variable $(X_1 + \cdots + X_n)/n$ is also Cauchy.

12. Let f be a bounded continuous function on $(0, \infty)$, having Laplace transform

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx, \quad \lambda \in (0, \infty).$$

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent exponential random variables, of parameter λ. Show that $\hat{f}$ has derivatives of all orders on $(0, \infty)$ and that, for all $n \in \mathbb{N}$, for some $C(\lambda, n) \neq 0$ independent of f, we have

$$(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n) \mathbb{E}\big(f(S_n)\big)$$

where $S_n = X_1 + \cdots + X_n$. Deduce that if $\hat{f} \equiv 0$ then also $f \equiv 0$.

13. For each $n \in \mathbb{N}$, there is a unique probability measure μ_n on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ such that $\mu_n(A) = \mu_n(UA)$ for all Borel sets A and all orthogonal $n \times n$ matrices U. Fix $k \in \mathbb{N}$ and, for $n \geq k$, let γ_n denote the probability measure on $\mathbb{R}^k$ which is the law of $\sqrt{n}(x^1, \dots, x^k)$ under μ_n. Show

(a) if $X \sim N(0, I_n)$ then $X/|X| \sim \mu_n$,

(b) if $(X_n : n \in \mathbb{N})$ is a sequence of independent $N(0,1)$ random variables and if $R_n = \sqrt{X_1^2 + \cdots + X_n^2}$ then $R_n / \sqrt{n} \to 1$ a.s.,

(c) γ_n converges weakly to the standard Gaussian distribution on $\mathbb{R}^k$ as $n \to \infty$.

14. Let $(E, \mathcal{E}, \mu)$ be a measure space and $\tau : E \to E$ a measure-preserving transformation. Show that $\mathcal{E}_\tau := \{A \in \mathcal{E} : \tau^{-1}(A) = A\}$ is a σ-algebra, and that a measurable function f is $\mathcal{E}_\tau$-measurable if and only if it is *invariant*, that is $f \circ \tau = f$.

15. Show that, if θ is an ergodic measure-preserving transformation and f is a θ-invariant function, then there exists a constant $c \in \mathbb{R}$ such that $f = c$ a.e..

16. For $x \in [0,1)$, set $\tau(x) = 2x \bmod 1$. Show that τ is a measure-preserving transformation of $([0,1), \mathcal{B}([0,1)), dx)$, and that τ is ergodic. Identify the invariant function $\overline{f}$ corresponding to each integrable function f.

17. Fix $a \in [0,1)$ and define, for $x \in [0,1)$, $\tau(x) = x + a \bmod 1$. Show that τ is also a measure-preserving transformation of $([0,1), \mathcal{B}([0,1)), dx)$. Determine for which values of a the transformation τ is ergodic. *Hint: you may use the fact that any integrable function f on $[0,1)$ whose Fourier coefficients all vanish must itself vanish a.e..* Identify, for all values of a, the invariant function $\overline{f}$ corresponding to an integrable function f.