PR (pr463@cam.ac.uk) Michaelmas 2025

Probability and Measure 3

1. Let C_n denote the nth approximation to the Cantor set C: thus $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n. Show that:
(a) C is uncountable and has Lebesgue measure 0;
(b) for all $x \in [0, 1]$, the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists;
(c) the function F is continuous on $[0, 1]$, with $F(0) = 0$ and $F(1) = 1$;
(d) for almost all $x \in [0, 1]$, F is differentiable at x with $F'(x) = 0$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.

2. We consider the unit circle of $\mathbb{R}^2$. We pick randomly a point on A on the circle, and then randomly another point B on the circle. We describe the random experiment using $\Omega = [0, 2\pi[^2$, $\mathcal{A} = \mathcal{B}([0, 2\pi[^2)$, $P(dw) = \frac{d\theta d\theta'}{4\pi^2}$.
(i) Compute the law of the r.v X=length of the chord $[AB]$.
(ii) Compute $P(X > \sqrt{3})$.

3. We consider the unit disc of $\mathbb{R}^2$. Put the following random experiment in equations: pick a point C in the unit disc, let A, B be the unique points on the unit disc such that C is the middle of $[AB]$, then compute the probability that the chord $[AB]$ has length $> \sqrt{3}$.

4(*). Let $\Omega = \{1, \dots, 6\}^{\mathbb{N}^*} = \{w = (w_1, \dots, w_6 \in \{1, \dots, 6\}\}$. We equip Ω with the σ-algebra $\mathcal{A}$ generated by the sets $A_{i_1,\dots,i_n} = \{w \in \Omega, w_1 = i_1, \dots, w_n = i_n\}$, $n \geq 1$.
(i) Show using the map $\phi(w) = \sum_{k=1}^{\infty} \frac{w_k - 1}{6^k}$ that Ω is uncontable.
(ii) Compute the image of $A_{i_1,\dots,i_n}$ by ϕ.
(iii) Study the injectivity of ϕ. Construct a probability measure on $\mathcal{A}$ such that $\forall n \geq 1$, $P(A_{i_1,\dots,i_n}) = \frac{1}{6^n}$.
(iv) Show that the map defined for $w \in \Omega$ by $X(w) = \inf\{j, w_j = 6\}$ is a discrete real r.v.
(v) Compute $P(X = k)$ and the probability of the set $\{X = \infty\}$. Is this set empty?

5. Let X be a random variable and let $1 \leq p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-p})$ as $\lambda \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-q})$ as $\lambda \to \infty$, then $X \in L^p(\mathbb{P})$.

6. The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

7. The moment generating function ϕ of a real-valued random variable X is defined by $\phi(\tau) = \mathbb{E}(e^{\tau X})$, $\tau \in \mathbb{R}$. Suppose that ϕ is finite on an open interval containing 0. Show that ϕ has derivatives of all orders at 0 and that X has finite moments of all orders given by

$$\mathbb{E}(X^n) = \left(\frac{d}{d\tau}\right)^n \Big|_{\tau=0} \phi(\tau).$$

8. Let $X = (X_1, \dots, X_d)$ be a r.v with value in $\mathbb{R}^d$ with $X_j \in L^2(\Omega, \mathcal{A}, P)$. Let the covariance matrix be $K_X = (\text{cov}(X_i, X_j))_{1 \le i,j \le d}$.
(i) Show that K_X is a symmetric positive matrix.
(ii) Let M be a (deterministic) $n \times d$ matrix. Let $Y = MX$, show that $K_Y = AK_XA^t$.

9. Let $X = (X_1, \dots, X_d)$ where X_j is a real valued square integrable r.v. Show that its characteristic function Φ_X is twice differentiable with

$$\Phi_X(\xi) = 1 + i\sum_{j=1}^{d} \xi_j E[X_j] - \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} \xi_j\xi_k E[X_jX_k] + o(|\xi|^2).$$

10. Let a symmetric definite positive square matrix $A \in M_d(\mathbb{R})$. Let X be a Gaussian variable with Gaussian law $p(x) = \frac{e^{-\frac{1}{2}(A^{-1}x|x)}}{\sqrt{(2\pi)^d \det(A)}}$, $x \in \mathbb{R}^d$. Compute the characteristic function of X.

11. Determine which of the following distributions on $\mathbb{R}$ have an integrable characteristic function: $\text{N}(\mu, \sigma^2)$, $\text{Bin}(N, p)$, $\text{Poisson}(\lambda)$, $\text{U}[0, 1]$.

12. (i) Let $\psi(x) = Ce^{-\frac{1}{1-x^2}}$ for $|x| < 1$ and $\psi(x) = 0$ otherwise, where C is a constant chosen so that $\int_{\mathbb{R}} \psi(x)dx = 1$. For $f \in L^1(\mathbb{R})$ of compact support, show that $f * \psi$ is C^∞ and has compact support.
(ii) Let $\sigma > 0$, $x \in \mathbb{R}$. $g_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. Show that $\forall \phi \in \mathcal{C}_b(\mathbb{R})$ (ie continuous bounded), $\forall x \in \mathbb{R}$, $\lim_{\sigma \to 0} g_\sigma \star \phi(x) = \phi(x)$.

13. Let $(X_n : n \in \mathbb{N})$ be independent $N(0, 1)$ random variables. Prove that

$$\limsup_n \left(X_n/\sqrt{2\log n}\right) = 1 \quad \text{a.s.}$$