PR (pr463@cam.ac.uk) Michaelmas 2025

Probability and Measure 2

1. Suppose that a simple function f has two representations $f = \sum_{k=1}^m a_k 1_{A_k} = \sum_{j=1}^n b_j 1_{B_j}$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \ldots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly B_δ for $\delta \in \{0,1\}^n$. Then set $f_{\varepsilon,\delta} = \sum_{k=1}^m \varepsilon_k a_k$ if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon,\delta} = 0$ otherwise.
(i) Show that, for any measure μ, $\sum_{k=1}^m a_k \mu(A_k) = \sum_{\varepsilon,\delta} f_{\varepsilon,\delta}\mu(A_\varepsilon \cap B_\delta)$.
(ii) Conclude that $\sum_{k=1}^m a_k \mu(A_k) = \sum_{j=1}^n b_j \mu(B_j)$.

2. Let μ and ν be finite Borel measures on $\mathbb{R}$. Let f be a continuous bounded function on $\mathbb{R}$. Show that f is integrable with respect to μ and ν. Show further that, if $\mu(f) = \nu(f)$ for all such f, then $\mu = \nu$.

3. (i) Consider the measure space $(\mathbb{N}, P(\mathbb{N}))$ equipped with the counting measure $\#$. Given $f : \mathbb{N} \to [0, +\infty]$ measurable, show that $\int_{\mathbb{N}} f d\# = \sum_{n=0}^{+\infty} f(n)$.
(ii) Let $u_{m,n}$ be a sequence of non negative terms such that $\forall n \geq 0$, the sequence $m \to u_{m,n}$ is non decreasing. Show that $\lim_{m\to+\infty} \sum_{n\geq 0} u_{m,n} \leq \sum_{n\geq 0} \lim_{m\to+\infty} u_{m,n}$.

4. Let $(f_k)_{k\geq 0}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$, integrable and such that $\sum_{k=0}^{+\infty} \int_{\mathbb{R}} |f_k| dx < +\infty$.
(i) For $n \geq 1$, let $A_n = \cap_{i\geq 1}\{x \in \mathbb{R}; \exists k \geq i, |f_k(x)| \geq \frac{1}{n}\}$. Show that A_n is negligible.
(ii) Show that $\lim_{k\to+\infty} f_k(x) = 0$ a.e.

5. (i) Compute $\lim_{n\to+\infty} \int_0^\infty \sin(e^x)/(1 + nx^2)dx$.
(ii) Compute $\lim_{n\to+\infty} \int_0^1 (n \cos x)/(1 + n^2 x^{\frac{3}{2}})dx$.
(iii) Compute $\lim_{n\to+\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{\frac{x}{2}} dx$.
(iv) Compute $\lim_{n\to+\infty} \int_0^n (\cos x)^n \left(1 - \frac{x}{n}\right)^n dx$.
(v) Let $f \in L^1(\mathbb{R})$, compute $\lim_{n\to+\infty} \int_{\mathbb{R}} \frac{f(x)}{1+n\sin^2 x} dx$.

6. Let u and v be differentiable functions on $\mathbb{R}$ with continuous derivatives u' and v'. Suppose that uv' and $u'v$ are integrable on $\mathbb{R}$ and $u(x)v(x) \to 0$ as $|x| \to \infty$. Show that

$$\int_{\mathbb{R}} u(x)v'(x)dx = -\int_{\mathbb{R}} u'(x)v(x)dx.$$

7. Show that the function $\sin x/x$ is not Lebesgue integrable over $[1, \infty)$ but that the integral $\int_1^N (\sin x/x)dx$ converges as $N \to \infty$.

8. Let $a > 0$. Let $B = \{x \in \mathbb{R}^d, \quad |x| \leq 1\}$ where $|x| = \left(x_1^2 + \cdots + x_d^2\right)^{\frac{1}{2}}$. Study the integrability in $L^1(B)$ and $L^1(\mathbb{R}^d \backslash B)$ of $f(x) = \frac{1}{|x|^a}$.

9. Let $(f_n)_{n \geq 0}$ be a sequence of measurable functions on X that converges a.e to a measurable function f.
(i) We assume $f_n \geq 0$ a.e. and $\lim_{n \to \infty} \int f_n dx = \ell < \infty$. Show that f is integrable and $\lim_{n \to +\infty} \|f_n - f\|_{L^1} = \ell - \int_X f dx$. Show that $\int f dx = \ell$ may fail.
(ii) Assume f is integrable and $\lim_{n \to +\infty} \|f_n\|_{L^1} = \|f\|_{L^1}$. Show that $f_n \to f$ in L^1.

10. (Brezis-Lieb lemma) Let $p \geq 1$, $f \in L^p(\mathbb{R}^d)$ and $(f_n)_{n \geq 0}$ a sequence of measurable functions from $\mathbb{R}^d \to \mathbb{R}$ such that $\lim_{n \to +\infty} f_n(x) = f(x)$ a.e $x \in \mathbb{R}^d$ and $\sup_{n \geq 1} \int_{\mathbb{R}^d} |f_n(x)|^p dx < +\infty$. We aim at proving

$$\lim_{n \to +\infty} \left[\int_{\mathbb{R}^d} |f_n(x)|^p dx - \int_{\mathbb{R}^d} |f(x) - f_n(x)|^p dx \right] = \int_{\mathbb{R}^d} |f(x)|^p dx.$$

(i) Prove the claim for $p = 1$.
(ii) Assume $p > 1$. Show that $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that $\forall (x, y) \in \mathbb{R}^2$, $|x + y|^p - |x|^p \leq \varepsilon |x|^p + C(\varepsilon)|y|^p$.
(iii) Pick $\varepsilon > 0$ and let $F_n = [||f_n|^p - |f_n - f|^p - |f|^p| - \varepsilon |f_n - f|^p]^+$. Compute $\lim_{n \to +\infty} \int_{\mathbb{R}^d} F_n(x) dx$ and conclude.

11. Let f, g be two measurable functions on $\mathbb{R}^d$.
(i) Show that the integral $f \star g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) dy$ is defined for a.e. x and that the function $f \star g$ defined in this way belongs to $L^1(\mathbb{R}^d)$ with $\|f \star g\|_{L^1} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$.
(ii) We assume in addition that $g \in \mathcal{C}^\infty(\mathbb{R}^d)$ and has compact support. Show that $f \star g \in \mathcal{C}^\infty(\mathbb{R}^d)$.
(iii) Assume f is continuous with compact support. Let ζ be a non negative $\mathcal{C}^\infty$ function with compact support and $\int \zeta(x) dx = 1$. Let $\zeta_\varepsilon = \frac{1}{\varepsilon^d} \zeta\left(\frac{x}{\varepsilon}\right)$, show that $f \star \zeta_\varepsilon \to f$ in L^1 as $\varepsilon \to 0$.
(iv) Conclude that the space of $\mathcal{C}^\infty$ compactly supported functions is dense in $L^1(\mathbb{R}^d)$.

12. (Riemann-Lebesgue lemma) For $f \in L^1(\mathbb{R}^d)$, we recall the definition of the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$.
(i) Show that $\hat{f}$ is a continuous function.
(ii) Let f be twice differentiable with compact support, compute the Fourier transform of $\partial_{x_i} f$ in terms of $\hat{f}$.
(iii) Prove that for all $f \in L^1(\mathbb{R}^d)$, $\lim_{\xi \to +\infty} \hat{f}(\xi) = 0$.

13. Let $(E, \mathcal{A}, \mu)$ be a measure space. We let $\bar{\mathcal{A}} = \sigma(\mathcal{A} \cup \mathcal{N})$ where $\mathcal{N}$ is the set of negligible sets.
(i) Let $\mathcal{C} = \{A \subset E, \exists B, B' \in \mathcal{A} \text{ with } B \subset A \subset B', \mu(B' \backslash B) = 0\}$. Show that $\mathcal{C}$ is a σ-algebra. Conclude that $\mathcal{C} = \bar{\mathcal{A}}$.
(ii) Show that for the Lebesgue measure on $\mathbb{R}$, $\overline{\mathcal{B}(\mathbb{R})} = \mathcal{M}(\lambda^*)$.
(iii) Show that if $f, g : \mathbb{R} \to \mathbb{R}$ with $f = g$ λ-a.e and f measurable for $\overline{\mathcal{B}(\mathbb{R})}$, then g is measurable for $\overline{\mathcal{B}(\mathbb{R})}$.