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Probability and Measure 1

1. Let E be a set and let $\mathcal{S}$ be a set of σ-algebras on E. Define

$$\mathcal{A}^* = \{A \subset E \;\text{ such that }\; A \in \mathcal{A} \quad \text{for all} \quad \mathcal{A} \in \mathcal{S}\}.$$

Show that $\mathcal{A}^*$ is a σ-algebra on E. Show, on the other hand, by example, that the union of two σ-algebras on the same set need not be a σ-algebra.

2. (i) Let $n \geq 1$, prove that $\mathcal{B}(\mathbb{R}^d)$ is generated by open balls.
(ii) Show that $\mathcal{B}(\mathbb{R})$ is generated by the family of intervals $\{(-\infty, a), a \in \mathbb{Q}\}$.

3. Let f, g be two continuous functions on $\mathbb{R}^d \to \mathbb{R}^{d'}$ that are equal a.e for Lebesgue's measure. Show that they are equal everywhere.

4. Let $(f_p)_{p\geq 0}$ be a sequence of continuous functions $f_p : \mathbb{R}^d \to \mathbb{R}$. Let A be the set of $x \in \mathbb{R}^d$ such that $\lim_{p\to+\infty} f_p(x) = 0$. Show that $A \in \mathcal{B}(\mathbb{R}^d)$.

5. (i) Consider a family $\mathcal{F}$ of functions $f : E \to (E', \mathcal{A}')$. Show that there exists a minimal σ-algebra $\mathcal{A}_{\mathcal{F}}$ such that all $f \in \mathcal{F}$ are measurable as functions from $(E, \mathcal{A}_{\mathcal{F}}) \to (E', \mathcal{A}')$.
(ii) Let $(E_1, \mathcal{A}_1)$ and $(E_2, \mathcal{A}_2)$ be two measurable spaces. Let the projection maps $\pi_1 : E_1 \times E_2 \to E_1$ and $\pi_2 : E_1 \times E_2 \to E_2$. Show that the product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the minimal σ-algebra which makes both π_1, π_2 measurable.
(iii) Let E_1, E_2 be two separable metric spaces (ie they admit a countable dense family). Show that $\mathcal{B}(E_1) \otimes \mathcal{B}(E_2) = \mathcal{B}(E_1 \times E_2)$.

6. Let $(E, \mathcal{A}, \mu)$ be a finite measure space. For any sequence of sets $(A_n)_{n\geq 0}$, $A_n \in \mathcal{A}$, define

$$\limsup A_n = \cap_{n\geq 0} \left(\cup_{k\geq n} A_k\right), \quad \liminf A_n = \cup_{n\geq 0} \left(\cap_{k\geq n} A_k\right).$$

Show that

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

7. Let $(X, \mathcal{A})$ be a measurable space equipped with a measure μ. Let $(A_n)_{n\geq 0}$ be a countable family of measurable sets. Let A_∞ be the set of $x \in X$ belonging to an infinite number of A_n. Let A_{tail} be the set of $x \in X$ belonging to all A_n except possibly a finite number of them.
(i) Show that A_∞, A_{tail} are measurable.
(ii) Prove that if A_n is non increasing (resp. non decreasing), then $A_{\text{tail}} = A_\infty = \cap_{n\geq 0} A_n$ (resp. $A_{\text{tail}} = A_\infty = \cup_{n\geq 0} A_n$).
(iii) Set $B_n = X \backslash A_n$. Compare B_∞, B_{tail} to A_{tail}, A_∞.
(iv) Show that $\lim_{n\to+\infty} \mu\left(\cap_{m\geq n} A_m\right) = \mu(A_{\text{tail}}) \leq \mu(A_\infty) \leq \lim_{n\to+\infty} \sum_{m\geq n} \mu(A_m)$.
(v) Prove the Borel Cantelli theorem: if $\sum_{n\geq 0} \mu(A_n) < +\infty$, then almost all point of $x \in X$ belong only to a finite number of A_n.
(vi) Show that if there exists n_0 such that $\mu\left(\cup_{m\geq n_0} A_m\right) < +\infty$, then $\mu(A_\infty) = \lim_{n\to+\infty} \mu\left(\cup_{m\geq n} A_m\right)$.

8. Prove that the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ is negligible for the two dimensional Lebesgue measure.

9. (i) Let f_1, f_2 be two measurable functions on a measurable space $(E, \mathcal{A})$. Show that $f_1 f_2$ and any linear combination of f_1, f_2 are also measurable.
(ii) Let $(f_n)_{n\geq 0}$ be a sequence of measurable functions on a measurable space $(E, \mathcal{A})$. Show that the following functions are also measurable: $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show that the set $\{x \in E, \quad f_n(x) \text{ has a limit as } n \to +\infty\}$ is measurable.
(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that f and f' are measurable.

10. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, let μ be a measure on $\mathcal{E}$ and $f : E \to G$ be a measurable function.
(i) Show that we can define a measure ν on $\mathcal{G}$ by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$. ν is called the image of μ by f and is noted $\nu = f_\star\mu = f(\mu)$.
(ii) Let $\nu = f(\mu)$, show that $g(\nu) = (g \circ f)(\mu)$ for all non-negative measurable functions g on G.
(iii) Take the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and compute $f_*\lambda$ for $f(x) = 0$. Is this measure σ-finite?
(iv) Take the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and compute $f_*\lambda$ for $f(x) = 2x$.