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Number Fields Examples

Michaelmas Term 2002

Sheet 3

1. For which primes p is the equation $x^2 + 13y^2 = p$ soluble in integers x, y?

2. Establish the following facts about the factorisation of principal ideals in $\mathbb{Q}(\sqrt{-d})$ where d is a positive square-free integer.

   (i) If d is composite and p is an odd prime divisor of d then $[p] = \wp^2$ where $\wp$ is not principal.

   (ii) If $d \equiv 1$ or $2 (\text{mod } 4)$ then $[2] = \wp^2$ where $\wp$ is not principal unless $d = 1$ or 2.

   (iii) If $d \equiv 7 (\text{mod } 8)$ then $[2] = \wp\overline{\wp}$ where $\wp$ is not principal unless $d = 7$.

   Hence show that if $\mathbb{Q}(\sqrt{-d})$ has class number 1 then either $d = 1$, 2 or 7, or d is prime and $d \equiv 3 (\text{mod } 8)$.

3. Show that $\mathbb{Q}(\sqrt{-d})$ has class number 1 for $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$.
   [Gauss conjectured that these are the only values and this has now been proved.]

4. (Tripos 94) State Dedekind's theorem on the ideal factorisation of rational primes in fields k with a power integral basis. Briefly outline the proof.
   Determine how the primes 2 and 5 factorise in $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/5}$.
   [It can be assumed that $\mathbb{Q}(\zeta)$ has an integral basis $1, \zeta, \zeta^2, \zeta^3$].

5. (Tripos 96) Define the ideal class group of an algebraic number field k. State a result involving the discriminant of k that implies that the group is finite.
   By Dedekind's theorem, or otherwise, factorise the primes 2 and 3 in the field $k = \mathbb{Q}(\sqrt{-23})$. Verify that the ideal equation

   $$[2, \omega][3, \omega] = [\omega]$$

   holds in k, where $\omega = \frac{1}{2}(1 + \sqrt{-23})$. Hence prove that k has class number 3.

6. (Tripos 98 - adapted) By Dedekind's theorem, or otherwise, factorize the primes 2 and 3 in the field $k = \mathbb{Q}(\sqrt{-17})$. Verify that 5 remains prime in k.
   Show that the ideal equation

   $$[\omega] = [2, \omega][3, \omega]^2$$

   holds in k, where $\omega = 1 + \sqrt{-17}$. Hence prove that the ideal class group of k is cyclic of order 4.

7. (Tripos 99 - adapted) Factorise the ideals [2], [3], and [5] in the ring of algebraic integers of the field $K = \mathbb{Q}(\sqrt{30})$. Using Minkowski's bound, determine the ideal class group of K.
   Find the fundamental unit of K and determine all solutions of the equations $x^2 - 30y^2 = \pm 5$ in integers x, y. Prove that there are in fact no solutions of $x^2 - 30y^2 = 5$.

8. (Tripos 00 - adapted) Let $f(x) = x^5 - x + 1$. Show that the discriminant of f is equal to $2869 = 19 \times 151$. Deduce that the ring of algebraic integers of $K = \mathbb{Q}(\alpha)$, where α is a zero of f, is $\mathcal{O}_K = \mathbb{Z}(\alpha)$. Using Minkowski's bound, determine the ideal class group of K.

9. (Tripos 00/01 - adapted) Using Minkowski's bound, determine the ideal class group of $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-11})$. Find all solutions in integers x, y of the diophantine equations
$$y^2 + 5 = x^3, \qquad y^2 + 11 = x^3.$$

10. (Tripos 02 - adapted) Let $K = \mathbb{Q}(\sqrt{35})$. By Dedekind's theorem, or otherwise, show that the ideal equations
$$2 = [2, \omega]^2, \quad 5 = [5, \omega]^2, \quad [\omega] = [2, \omega][5, \omega]$$
hold in K, where $\omega = 5 + \sqrt{35}$. Deduce that K has class number 2.
    Verify that $1 + \omega$ is the fundamental unit in K. Hence show that the complete solution in integers x, y of the equation $x^2 - 35y^2 = -10$ is given by
$$x + \sqrt{35}y = \pm\omega(1 + \omega)^n \quad (n = 0, \pm 1, \pm 2, \ldots).$$
    Calculate the particular solution x, y for $n = 1$.

11. (Tripos 92) Factorise [2] in the ring of integers of $\mathbb{Q}(\sqrt{65})$. Show that the primes dividing [2] are not principal. Find the ideal class group of $\mathbb{Q}(\sqrt{65})$.
    Describe all integer solutions of $X^2 - 65Y^2 = 40$.

12. (Tripos 93) Factorise the ideals [2], [5], $[1 + \sqrt{-26}]$ and $[2 + \sqrt{-26}]$ in the ring of integers of $\mathbb{Q}(\sqrt{-26})$. Find the ideal class group of the ring of integers of $\mathbb{Q}(\sqrt{-26})$.

13. (Tripos 94) Factorise the ideals [2], [3] and $[2 + \sqrt{-14}]$ in the ring of integers of $\mathbb{Q}(\sqrt{-14})$. Find the ideal class group of the ring of integers of $\mathbb{Q}(\sqrt{-14})$.

14. Find the class group of $\mathbb{Q}(\alpha)$, where α is
$$\sqrt{-6}, \sqrt{10}, \sqrt{-29}, \sqrt[3]{2}, e^{2\pi i/5}.$$

15. Let $\alpha = e^{2\pi i/7} + e^{-2\pi i/7}$. Show that the minimum polynomial of α is $f(x) = x^3 + x^2 - 2x - 1$. Determine the discriminant of f and the ideal class group of $\mathbb{Q}(\alpha)$.