Example Sheet 1

1. Let d be any integer > 1. Show that the only units of the ring $\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$ are ± 1.

2. Prove that the ring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ has infinitely many units.

3. Is $\dfrac{3+2\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

4. Let $K = \mathbb{Q}(\theta)$, where θ is any root in $\mathbb{C}$ of $X^3 - 2X + 6$. Prove that K has degree 3 over $\mathbb{Q}$, and compute
(i) $N_{K/\mathbb{Q}}(n - \theta)$ $(n = 1, 2, 3)$, (ii) $N_{K/\mathbb{Q}}(1 - \theta^2)$, (iii) $Tr_{K/\mathbb{Q}}(n - \theta)$ $(n = 1, 2, 3)$, (iv) $Tr_{K/\mathbb{Q}}(1 - \theta^2)$, (v) $Tr_{K/\mathbb{Q}}(1 - \theta^3)$, (vi) $N_{K/\mathbb{Q}}(1 - \theta^3)$.

5. Let p be an odd prime number, and let $\zeta = e^{\frac{2\pi i}{p}}$. Determine the degree over $\mathbb{Q}$ of $K = \mathbb{Q}(\zeta)$, and calculate
(i) $N_{K/\mathbb{Q}}(1 - \zeta^j)$, and (ii) $Tr_{K/\mathbb{Q}}(1 - \zeta^j)$ $(j = 1, \ldots, p-1)$.

6. Show that $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is an integral basis of $\mathbb{Q}(\sqrt[3]{2})$.

7. Show that $\{1, \theta, \frac{1}{2}(\theta + \theta^2)\}$ is an integral basis of $\mathbb{Q}(\theta)$, where θ is a root of $X^3 - X - 4$.

8. Show that

$$54 = 2 \cdot 3^3 = \frac{13 + \sqrt{-47}}{2} \cdot \frac{13 - \sqrt{-47}}{2}$$

are two essentially different decompositions of 54 into irreducible elements in the ring of integers of $\mathbb{Q}(\sqrt{-47})$.