Number Fields: Example Sheet 2 of 3

1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $\mathcal{O}_K$. Determine the factorisations into prime ideals of $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$ in terms of those for $\mathfrak{a}$ and $\mathfrak{b}$. Show that if $\mathfrak{a}+\mathfrak{b}=\mathcal{O}_K$ then $\mathfrak{a}\cap\mathfrak{b}=\mathfrak{a}\mathfrak{b}$ and there is an isomorphism of rings $\mathcal{O}_K/\mathfrak{a}\mathfrak{b}\cong\mathcal{O}_K/\mathfrak{a}\times\mathcal{O}_K/\mathfrak{b}$.

2. Let $K=\mathbb{Q}(\sqrt{-5})$. Show by computing norms, or otherwise, that $\mathfrak{p}=(2,1+\sqrt{-5})$, $\mathfrak{q}_1=(7,3+\sqrt{-5})$ and $\mathfrak{q}_2=(7,3-\sqrt{-5})$ are prime ideals in $\mathcal{O}_K$. Which (if any) of the ideals $\mathfrak{p},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{p}^2,\mathfrak{p}\mathfrak{q}_1,\mathfrak{p}\mathfrak{q}_2$ and $\mathfrak{q}_1\mathfrak{q}_2$ are principal? Factor the principal ideal $(9+11\sqrt{-5})$ as a product of prime ideals.

3. Let $\mathfrak{a}\subset\mathcal{O}_K$ be a non-zero ideal, and m the least positive integer in $\mathfrak{a}$. Prove that m and $N\mathfrak{a}$ have the same prime factors.

4. Let $K=\mathbb{Q}(\sqrt{35})$ and $\omega=5+\sqrt{35}$. Verify the ideal equations $(2)=(2,\omega)^2$, $(5)=(5,\omega)^2$ and $(\omega)=(2,\omega)(5,\omega)$. Show that the class group of K contains an element of order 2. Find all ideals of norm dividing 100 and determine which are principal.

5. Let p be an odd prime and $K=\mathbb{Q}(\zeta_p)$ where ζ_p is a primitive pth root of unity. Determine $[K:\mathbb{Q}]$. Calculate $N_{K/\mathbb{Q}}(\pi)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\pi)$ where $\pi=1-\zeta_p$.
   (i) By considering traces $\mathrm{Tr}_{K/\mathbb{Q}}(\zeta_p^j\alpha)$ show that $\mathbb{Z}[\zeta_p]\subset\mathcal{O}_K\subset\frac{1}{p}\mathbb{Z}[\zeta_p]$.
   (ii) Show that $(1-\zeta_p^r)/(1-\zeta_p^s)$ is a unit for all $r,s\in\mathbb{Z}$ coprime to p, and that $\pi^{p-1}=up$ where u is a unit.
   (iii) Prove that the natural map $\mathbb{Z}\to\mathcal{O}_K/(\pi)$ is surjective. Deduce that for any $\alpha\in\mathcal{O}_K$ and $m\geq 1$ there exist $a_0,\ldots,a_{m-1}\in\mathbb{Z}$ such that
   $$\alpha\equiv a_0+a_1\pi+\ldots+a_{m-1}\pi^{m-1}\pmod{\pi^m\mathcal{O}_K}.$$
   (iv) Deduce that $\mathcal{O}_K=\mathbb{Z}[\zeta_p]$.

6. Let $K=\mathbb{Q}(\sqrt{-d})$ where d is a positive square-free integer. Establish the following facts about the factorisation of principal ideals in $\mathcal{O}_K$.
   (i) If d is composite and p is an odd prime divisor of d then $(p)=\mathfrak{p}^2$ where $\mathfrak{p}$ is not principal.
   (ii) If $d\equiv 1$ or $2\pmod 4$ then $(2)=\mathfrak{p}^2$ where $\mathfrak{p}$ is not principal unless $d=1$ or 2.
   (iii) If $d\equiv 7\pmod 8$ then $(2)=\mathfrak{p}\overline{\mathfrak{p}}$ where $\mathfrak{p}$ is not principal unless $d=7$.

   Deduce that if K has class number 1 then either $d=1$, 2 or 7, or d is prime and $d\equiv 3\pmod 8$.

7. Let $K=\mathbb{Q}(\sqrt{-m})$ where $m>0$ is the product of distinct primes $p_1,\ldots,p_k$. Show that $(p_i)=\mathfrak{p}_i^2$ where $\mathfrak{p}_i=(p_i,\sqrt{-m})$. Show that just two of the ideals $\prod\mathfrak{p}_i^{r_i}$ with $r_i\in\{0,1\}$ are principal. Deduce that the class group Cl_K contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{k-1}$. [*If you like, just do the case* $m\not\equiv 3\pmod 4$.]

8. Let $K = \mathbb{Q}(\theta)$ where θ is a root of $X^3 - 4X + 7$. Determine the ring of integers and discriminant of K. Determine the factorisation into prime ideals of $p\mathcal{O}_K$ for $p = 2, 3, 5, 7, 11$. Find all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$ with $N\mathfrak{a} \leq 11$.

9. Let $K = \mathbb{Q}(\alpha)$ where α is a root of $f(X) = X^3 + X^2 - 2X + 8$. [*This polynomial is irreducible over* $\mathbb{Q}$ *and has discriminant* -4×503.]

   (i) Show that $\beta = 4/\alpha \in \mathcal{O}_K$ and $\beta \notin \mathbb{Z}[\alpha]$. Deduce that $\mathcal{O}_K = \mathbb{Z}[\alpha, \beta]$.

   (ii) Show that there is an isomorphism of rings $\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$. Deduce that 2 splits completely in K.

   (iii) Use Dedekind's criterion to show that $\mathcal{O}_K \neq \mathbb{Z}[\theta]$ for any θ.

10. (i) Let $\mathfrak{a} \subset \mathcal{O}_K$ be a non-zero ideal. Show that every ideal in the ring $\mathcal{O}_K/\mathfrak{a}$ is principal. [*Hint: Use Question 1 to reduce to the case* $\mathfrak{a}$ *is a prime power.*]

    (ii) Deduce that every ideal in $\mathcal{O}_K$ can be generated by 2 elements.

11. Show that $\mathbb{Q}(\sqrt{-d})$ has class number 1 for d = 1, 2, 3, 7, 11, 19, 43, 67, 163.

And some extra questions.

12. The zeta function $\zeta_K(s)$ of a number field K is the infinite sum $\zeta_K(s) = \sum_{\mathfrak{a} \leq \mathcal{O}_K} N(\mathfrak{a})^{-s}$, for $s \in \mathbb{C}$. Show that this factors 'formally' as an infinite product, $\zeta_K(s) = \prod_{\mathfrak{p} \leq \mathcal{O}_K, \mathfrak{p}\, prime}(1 - N(\mathfrak{p})^{-s})^{-1}$. where the product has a term for each prime ideal of $\mathcal{O}_K$.

    (ii) Now let $K = \mathbb{Q}(\sqrt{d})$, where $d \neq 0, 1$ and d is square free. Show the zeta function factors 'formally', $\zeta_K(s) = \zeta_{\mathbb{Q}}(s).L(\chi, s)$ where $L(\chi, s) = \prod_{p\, prime}(1 - \chi(p)p^{-s})^{-1}$, for an explicit function χ, which you should determine in terms of how the ideal (p) factorises in $\mathcal{O}_K$.

    'Formally' means the terms match up, i.e. you do not need to discuss convergence for this question. We will study convergence of these sums in the last 3 lectures.

13. For $\mathfrak{a}$ an ideal in $\mathcal{O}_K$ let $\phi(\mathfrak{a}) = |(\mathcal{O}_K/\mathfrak{a})^*|$. Show that $\phi(\mathfrak{a}) = N(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}}(1 - \frac{1}{N\mathfrak{p}})$.

14. Let $K = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(f(x))$, where $f(x)$ is the minimum polynomial of α. The trace form on K defines a bilinear form on the real vector space $\mathbb{R}[x]/(f(x))$. Show that the signature of this form is $(r + s, s)$, where $r + 2s = n$, r is the number of embeddings of K into $\mathbb{R}$.

15. Prove Stickelberger's criterion, that $D_K \equiv 0, 1 \pmod 4$. [*Hint: Start by writing* $D_K = (P - N)^2 = (P + N)^2 - 4PN$ *where* P *is a sum over even permutations and* N *is a sum over odd permutations. Then show that* $P + N, PN \in \mathbb{Z}$.]
    Hence compute the ring of integers of $\mathbb{Q}[X]/(f(X))$ where $f(X) = X^3 - X + 2$.

16. Let $B_{r,s}(t) = \{(y_1, \ldots, y_r, z_1, \ldots, z_s) \in \mathbb{R}^r \times \mathbb{C}^s \mid \sum |y_i| + 2\sum |z_j| \leq t\}$. Show that $volB_{r+1,s}(t) = \int_{-t}^{t} volB_{r,s}(t - |y|)dy$, and $volB_{r,s+1}(t) = \int\int_{|z| \leq t/2} volB_{r,s}(t - 2|z|)$.

    Hence show by induction that $volB_{r,s}(t) = 2^r(\frac{\pi}{2})^s \frac{t^n}{n!}$. [You should do the second integral by choosing polar coordinates, $z = re^{i\theta}$.]